Self-consistent Effective-medium Approximation for Strong-field Magneto-transport in a Composite Medium

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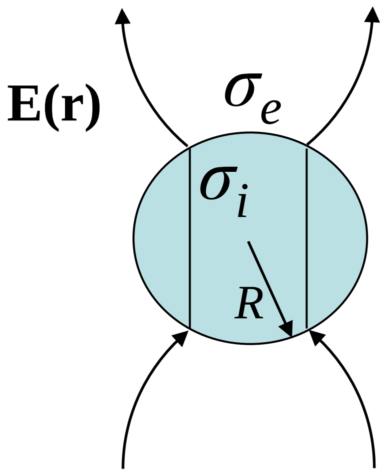
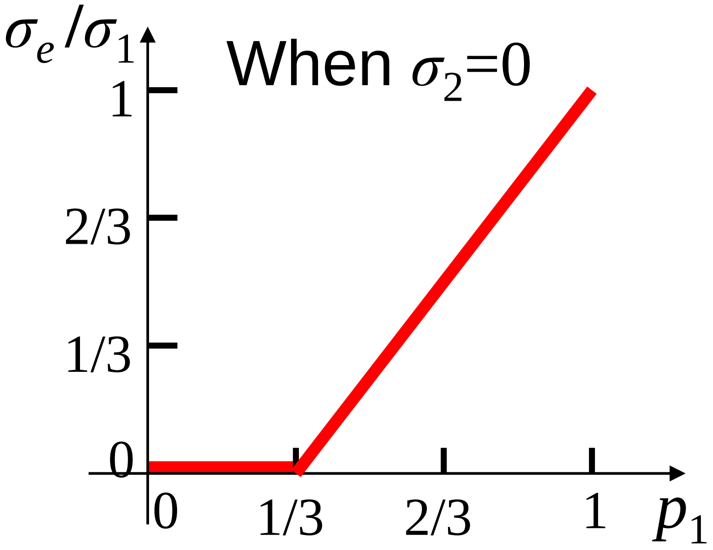
Introduction

- Except in a composite with a flat laminar micro-structure, electric field and current distribution are very complex, even when all constituents are simple scalar conductors.
- When a magnetic field is present the complexity increases: The conductivity is now a non-symmetric tensor.

Self-consistent-effective-medium-approximation (SEMA) (Bruggeman 1935, Landauer 1952)

- This is symmetric in the two constituents and has a percolation or conductivity threshold, which is a critical point.
- In other respects it is unsatisfactory: Critical exponents have wrong values.
- The approximation is uncontrolled and cannot be improved.





$$0 = p_1 \frac{\sigma_1 - \sigma_e}{\sigma_1 + 2\sigma_e} + p_2 \frac{\sigma_2 - \sigma_e}{\sigma_2 + 2\sigma_e}$$

$$\Rightarrow 4\sigma_e = \sigma_1(3p_1 - 1) + \sigma_2(3p_2 - 1) + \sqrt{\left[\sigma_1(3p_1 - 1) + \sigma_2(3p_2 - 1)\right]^2 + 8\sigma_1\sigma_2}$$

SEMA for Non-Symmetric Conductivity Tensors

$$\mathbf{E}_{\text{internal}} = \hat{\gamma} \cdot \mathbf{E}_0 \quad \text{for any ellipsoidal inclusion}$$

$$\left(\frac{1}{\hat{\gamma}}\right)_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{\omega} \frac{n_{\alpha\omega}\left(\sigma_{\omega\beta}^{(\text{host})} - \sigma_{\omega\beta}^{(\text{inclusion})}\right)}{\left(\sigma_{\alpha\alpha}^{(\text{host})}\sigma_{\omega\omega}^{(\text{host})}\right)^{1/2}}, \quad \text{where } \hat{n} \text{ is the depolarization tensor.}$$

The change in the volume averaged current density $\langle \mathbf{J} \rangle$ by a single inclusion is:

$$\langle \mathbf{J} \rangle - \hat{\sigma}_{\text{host}} \langle \mathbf{E} \rangle = \frac{V_{\text{incl}}}{V}\left(\hat{\sigma}_{\text{incl}} - \hat{\sigma}_{\text{host}}\right) \cdot \mathbf{E}_{\text{internal}} = \frac{V_{\text{incl}}}{V}\left(\hat{\sigma}_{\text{incl}} - \hat{\sigma}_{\text{host}}\right) \cdot \hat{\gamma} \cdot \mathbf{E}_0$$

Averaging this change over the different types of inclusions j and equating the result to 0 identifies $\hat{\sigma}_{\text{host}}$ as the macroscopic or bulk effective $\hat{\sigma}_e$:

$$0 = \sum_j p_j \left(\hat{\sigma}_j - \hat{\sigma}_e\right) \cdot \hat{\gamma}\left(\hat{\sigma}_j, \hat{\sigma}_e\right)$$

Our notation

$$\mathbf{B} \parallel z, \text{ isotropic conductor:}$$

$$\hat{\rho} \equiv \begin{pmatrix} \rho_\perp & -\rho_H & 0 \\ \rho_H & \rho_\perp & 0 \\ 0 & 0 & \rho_\parallel \end{pmatrix} = \rho_0 \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

- $\rho_H = \rho_0 \beta$ is the Hall resistivity
- $\rho_\perp = \rho_0 \alpha$ is the transverse Ohmic resistivity
- $\rho_\parallel = \rho_0 \lambda$ is the longitudinal Ohmic resistivity
- $H \equiv \beta/\alpha = \omega_c \tau = \mu|\mathbf{B}| = \rho_H/\rho_\perp$ is the Hall-to-transverse-Ohmic resistivity ratio
- In a free electron gas conductor $\lambda = \alpha$ and is independent of $\mathbf{B}$,
  while $\beta \propto |\mathbf{B}|$.

High Field Results

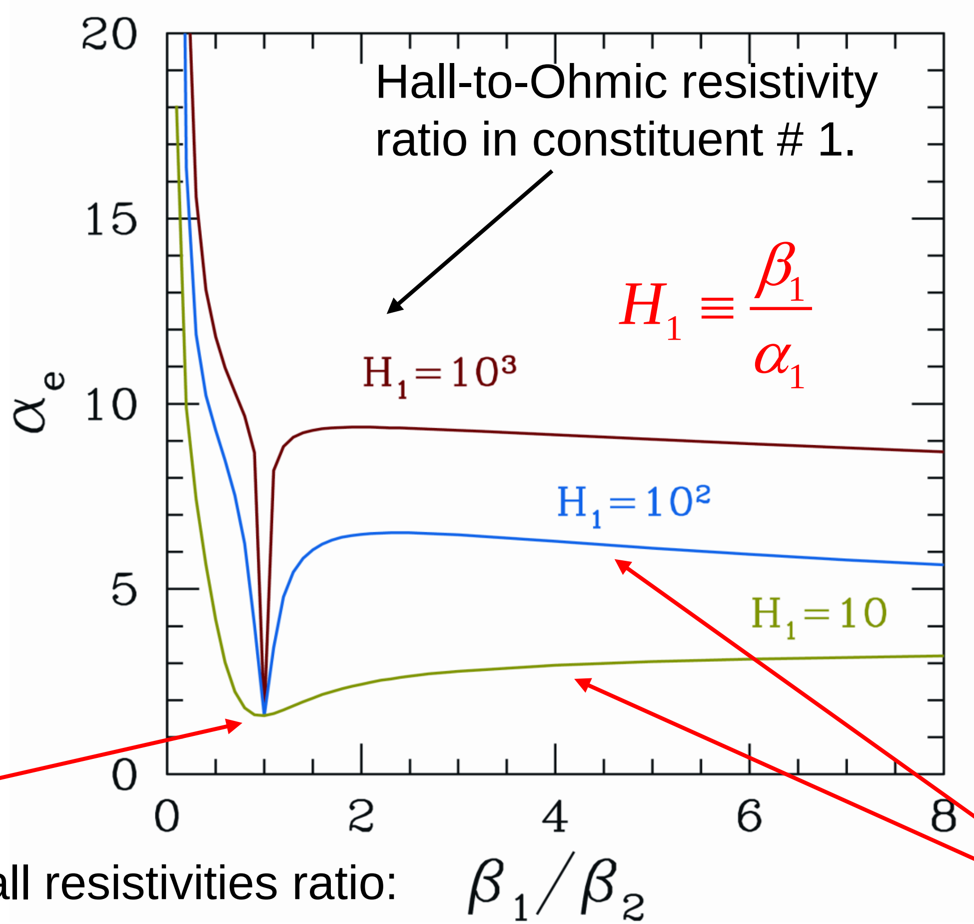
- Three coupled equations are found for the three resistivity parameters α_e, β_e, λ_e. They are explicit but horribly nonlinear, and include transcendental functions of these unknown parameters. They can be solved numerically.
- When $\beta_i >> \alpha_i$ and $\beta_i >> \lambda_i$, an asymptotic expansion leads to $\lambda_e \cong \langle \lambda \rangle$ and to closed form results for α_e, β_e which exhibit two new critical points.
- A microstructure-independent critical point is found when $\beta_1/\beta_2 = 1$: At that point we find $\alpha_e \cong \langle \alpha \rangle$. Otherwise we find $\alpha_e \propto \beta^{2/3}$.
- A microstructure-dependent critical point is found when $\beta_1/\beta_2 < 0$ and

$$p_1 = p_{1c} \equiv \frac{1}{1 - \beta_2 / \beta_1}.$$

Microstructure-independent Critical Point

Two conducting constituents; volume fraction: p_1=0.6;
Ohmic resistivity components are comparable and fixed, i.e.,
not metal/insulator mixture

Transverse Ohmic resistivity:



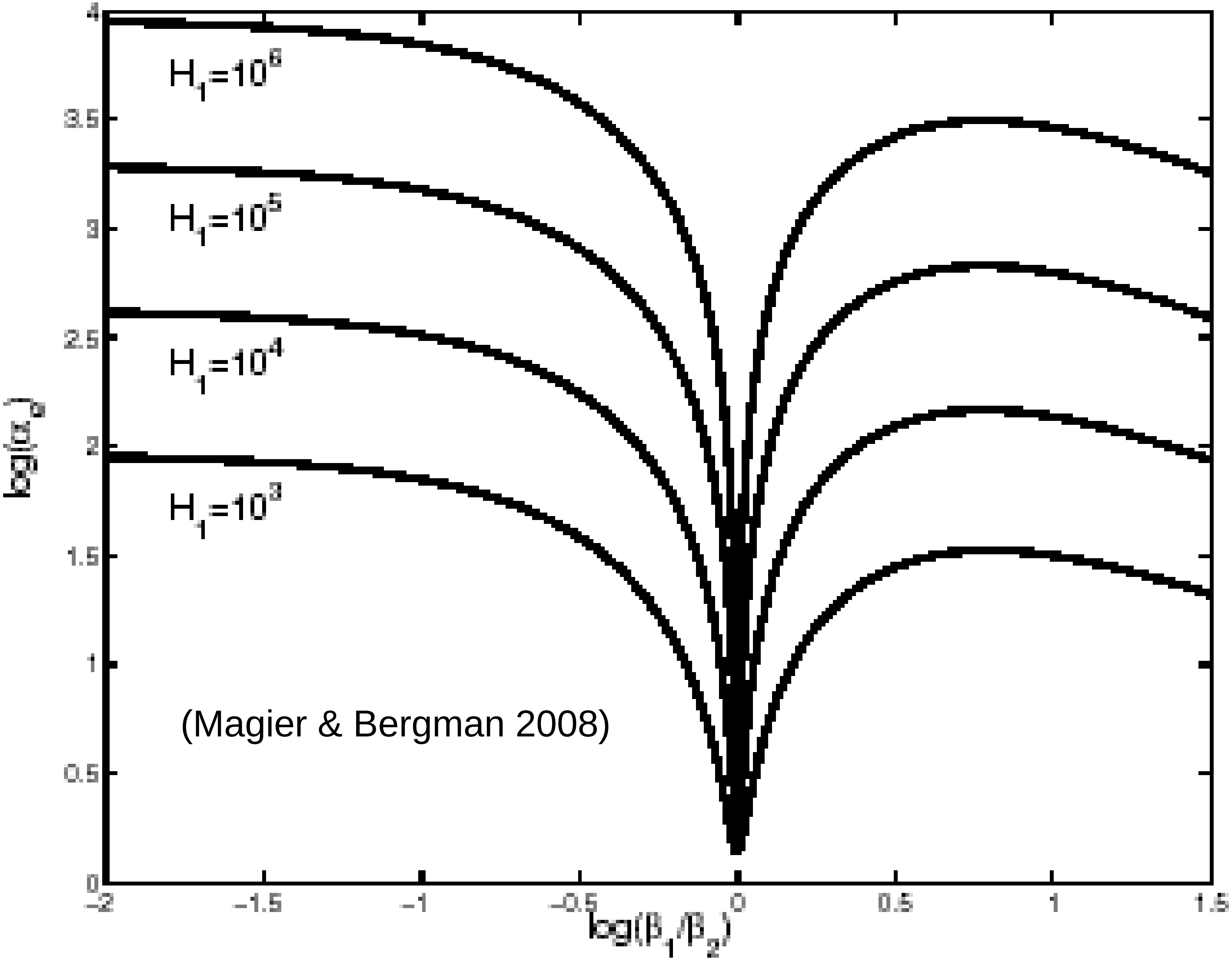
$\alpha_e \cong \langle \alpha \rangle$

Hall resistivities ratio: β_1/β_2

$\alpha_e \propto \beta^{2/3}$

(Magier & Bergman 2008)

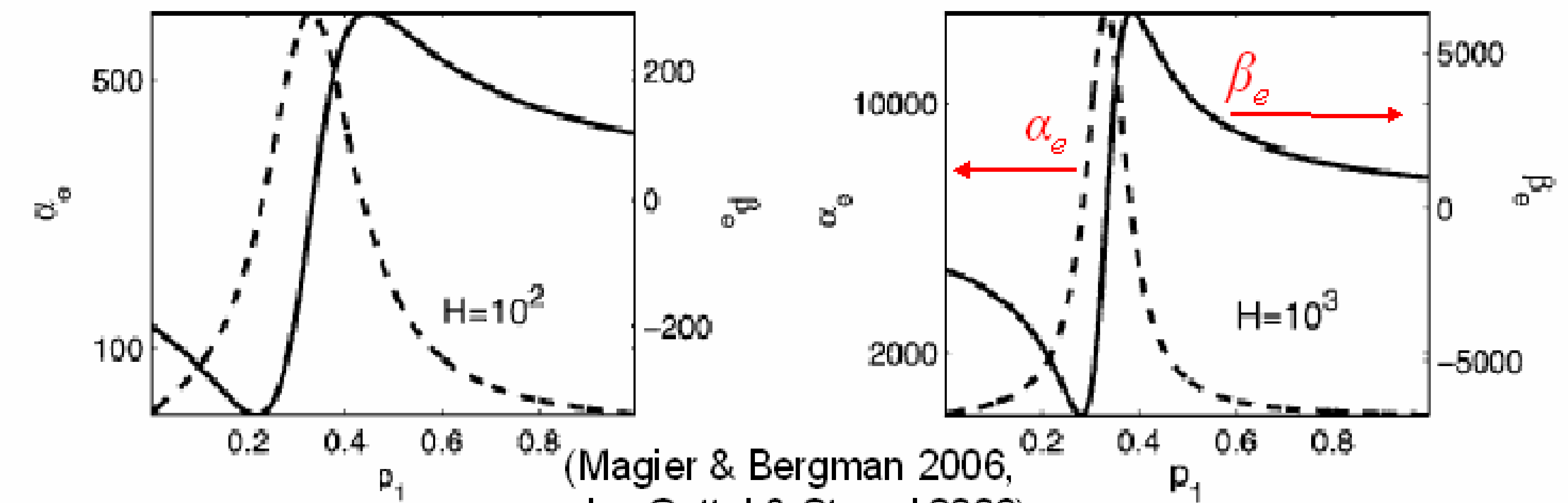
$\log \alpha_e$ vs. $\log(\beta_1/\beta_2)$ for $p_1 = 0.6$ and different values of H_1



Microstructure-dependent critical point

α_e and β_e vs. p_1 for $\beta_2/\beta_1 = -2$ and different values of H_1

$1/p_{1c} \approx 1 - \beta_2/\beta_1 = 3$



(Magier & Bergman 2006, also Guttal & Stroud 2006)

Away from $p_1 = p_{1c}$ we have $\dfrac{1}{\beta_e} \cong \left\langle \dfrac{1}{\beta} \right\rangle.$

This is corroborated by exact asymptotics!

Away from p_{1c} we also have $\alpha_e \propto \beta^{2/3}.$

For p_1 near p_{1c} we have $\beta_e \propto \left(p_1 - p_{1c}\right)\beta^{5/3}$ and $\alpha_e \propto \beta^{4/3}.$

Conclusions

- The self-consistent effective-medium approximation (SEMA) leads to explicit coupled nonlinear equations for elements of the macroscopic resistivity tensor.
- When the Hall resistivity is much greater than the Ohmic resistivities in the constituents an asymptotic solution of those equations can be obtained in closed form.
- An exact asymptotic expansion was motivated by those SEMA results. It corroborates the leading order SEMA results.
- New critical points were first found using SEMA.
- SEMA is also a useful tool for developing physical insight and understanding for the behavior and response of a complex composite medium. It should not be discarded!