Resonant Eigenfunction Delocalization of Random Schrödinger Operators on Tree Graph

1. *Unbounded potentials: Extended States in a Lifshitz Tail Regime*
2. *Bounded potentials: Absence of Mobility Edge at Weak Disorder*

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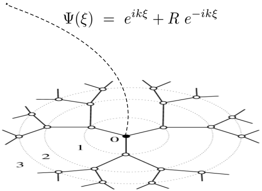
Statistical Mechanics Day (IV)
Weizmann Institute Science
June 22, 2011

The random Hamiltonian

- $\boxed{H_\lambda(\omega) := T + \lambda V(x;\omega)}$

- on the homogeneous tree graph of degree $K+1$; **Bethe lattice** $\mathbb{B}$.



Hopping term: $(T\psi)(x) := -\sum_{\text{dist}(x,y)=1} \psi(y)$

Random potential: $V(x;\cdot)$, $x \in \mathbb{B}$, i.i.d. random variables,

$\mathbb{P}(V(0) \in dv) = \varrho(v)\, dv$, ϱ bounded, piecewise monotone & piecewse cont.

In this talk, special emphasis on two cases:

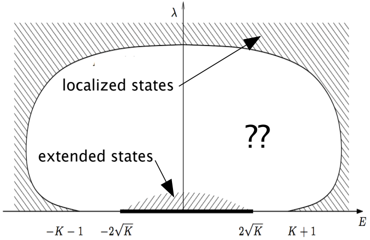
i. supp $\varrho = \mathbb{R}$ (e.g. $V(x)$ Gaussian or Cauchy)

ii. supp $\varrho = [-1, 1]$ (bounded potential, $|V(x;\omega)| \leq 1$)

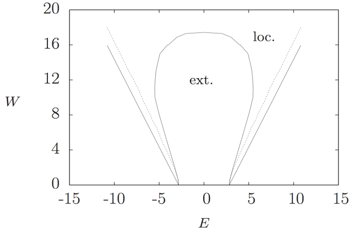
Alternative form:

$$\boxed{\mathcal{H}(\omega) = -t \sum_{\langle x,y\rangle \subset \mathbb{B}} a_x^\dagger a_y + \frac{W}{2} \sum_{x \in \mathbb{B}} V(x;\omega)\, a_x^\dagger a_x}$$

Previous results and expected phase diagrams

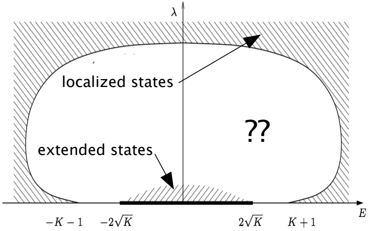


phase diagram for
unbounded potential



expected mobility edge
for a bounded potential

- Among the earliest studied models of the Anderson localization And. '58
  Abou-Chacra/Anderson/Thouless '73, Abou-Chacra/Thouless '74
- Motivation: Relatively more accessible compared to $\mathbb{Z}^d$.
  Self-consistent approach to localization becomes exact ($\neq$ solvable !).
- Renewed interest as a model for the configuration space of systems with many particles Altshuler/Gefen/Kamenev/Levitov '97 , (cf. Pal/Huse'11)
- Numerical work: Miller/Derrida '94, Biroli/Semerjian/Tarzia '10
- Rigorous results: *next page*



$$\boxed{H = T + \lambda V}$$

supp $\varrho = \mathbb{R}$

+ certain regularity conditions

- Spectrum of the Laplacian on $\ell^2(\mathbb{B})$: $\sigma(T) = \left[-2\sqrt{K}, 2\sqrt{K}\right]$

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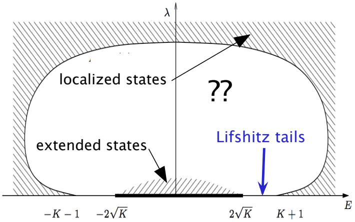
1. pp (loc.) spectrum at strong disorder: Aizenman/Molchanov '93
   and at large energies Aiz. '94
2. ac (ext.) spectrum for weak disorder at energies within $\sigma(T)$ Klein '94
   Aiz./Sims/Warzel '05, Froese/Hasler/Spitzer '06
3. Lifshitz tails: For $\lambda \to 0$, the mean density of states (ρ_{DOS}) at energies outside $\sigma(T)$ vanishes faster than any power of λ Miller/Derrida '94

The long open puzzle



supp $\varrho = \mathbb{R}$

$\int_{\mathbb{R}} v\, \varrho(v)\, dv = 0$

Question: Where is the edge of the localization regime, in particular, at weak disorder?

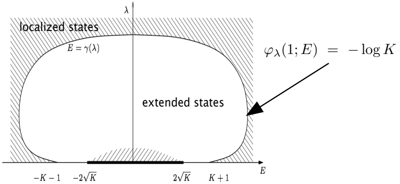
Note: for energies E outside $\sigma(T) = [-2\sqrt{K}, 2\sqrt{K}]$, the mDOS ($\rho_{\mathrm{DOS}}$) vanishes at weak disorder to all orders in perturbation theory (Lifshitz tail regime)

E.g., in case of the Gaussian random potential, for $E \notin \sigma(T)$:

$$\rho_{\mathrm{DOS}}(E) \approx \exp\left(-C(E)/\lambda^2\right) \quad \text{as } \lambda \downarrow 0.$$

A somewhat surprising answer

Theorem: In the case of unbounded random potential (supp $\rho = \mathbb{R}$, etc.), for $\lambda > 0$ the ac spectrum immediately extends up to $E = \pm(K+1)$, in particular, into the regime of Lifshitz tails.



More can be said in terms of the Green function

$$G(0,x;E) := \left\langle \delta_0, (H - E - i0)^{-1} \delta_x \right\rangle$$

and its moment generating function

$$\text{for } s < 1: \qquad \varphi_\lambda(s;E) := \lim_{|x|\to\infty} \frac{\log \mathbb{E}\left[|G_\lambda(0,x;E+i0)|^s\right]}{|x|}$$

$$\text{for } s = 1: \qquad \varphi_\lambda(1;E) := \lim_{s\nearrow 1} \varphi_\lambda(s;E)$$

Heuristics

The mechanism at work here: fluctuation enabled resonant tunneling:

States which locally appear to be localized have arbitrarily close energy gaps (ΔE)

with other states (at distances R), to which the tunneling amplitudes are

exponentially small (as $\approx e^{-L_\lambda(E)R}$) .

Mixing between two levels occurs if $\boxed{\Delta E \ll e^{-L(\lambda(E)R}}$.

Since the volume grows exponentially fast (as K^R),

extended states will form provided

$$\boxed{L_\lambda(E) < \log K}$$

Essential enabling conditions:

- local fluctuations in the self energy
- the exponential growth of the configuration space volume

(Almost) complementary criteria for pp and ac spectra

Assumptions: $\varrho(V) > 0$ on $\mathbb{R}$, $\int |v|^\tau \varrho(v)dv < \infty$ for some $\tau > 0$, etc.

Theorem 1 (localization [Aiz./Molchanov '93, Aiz. '94])

- *If for all (or a.e.) energies E in some interval $I \subset \mathbb{R}$*

$$\boxed{\varphi_\lambda(1; E) < -\log K} \tag{1}$$

then $H(\omega)$ has only pure point (localized) spectrum in that interval.

- *Furthermore, at weak disorder (1) holds for energies $|E| > (K+1)$.*

The new, *complementary*, statement:

Theorem 2 (delocalization [Aiz./Warzel '10, '11])

- Under the above assumptions on ρ, at energies at which

$$\boxed{\varphi_\lambda(1; E) > -\log K} \tag{2}$$

one has: $\boxed{\mathrm{Im}\, G(x, x; E + i0) \; > \; 0\,;}$

- ($\Longrightarrow$) if (2) holds for a positive measure of energies $E \in I$, then $H(\omega)$ has absolutely continuous (delocalized) spectrum in that interval.

Added by M. Shamis: One may conclude from Thm. 2 that if (2) holds for almost every $E \in I$ then then $H(\omega)$ has only ac spectrum in I.

Implications for the phase diagram

Assuming:

i. the function $\varphi_\lambda(1;E)$, and/or just the related Lyapunov exponent, are continuous in (E,λ) (there are gaps here in the mathematical theory),

ii. the equality $\boxed{\varphi_\lambda(1;E) = -\log K}$ holds only along a simple curve,

one may conclude:

1. the random operator has a (simple) mobility edge which converges to $|\mathbb{E}| = \pm(K+1)$ (and not to the edge of the $\lambda = 0$ spectrum $\sigma(T)$).

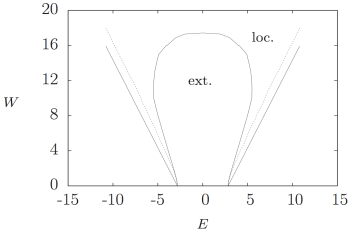
2. up to the mobility edge the random operator has (*a.s.*) purely extended states; beyond it only pure point spectrum, with dynamical localization and all that.

For bounded potentials, Theorem 2 yields another surprise:

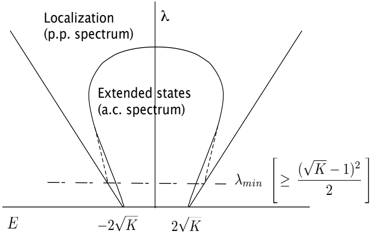
There is no such mobility edge at weak disorder !

the (generally) expected phase diagram needs to be corrected.

Bounded potentials: absence of (a simple) mobility edge at weak disorder



prev. published numerical results



sketch of the corrected phase diagram

Theorem 3

For H_λ as in Theorem 2, with a bounded random potential of regular distribution, for any

$$\lambda < (\sqrt{K} - 1)^2/2$$

the random operator has absolutely continuous spectrum (delocalizes states) at energies arbitrarily close to the edge of its spectrum

(the edge being at $E_\lambda = \pm(2\sqrt{K} + \lambda)$).

References (to the new results presented here):

1. M. Aizenman, S. Warzel, "Extended States in a Lifshitz Tail Regime for Random Schrödinger Operators on Trees", Phys. Rev. Lett. **106**, 136804 (2011).

2. M. Aizenman, S. Warzel, "Resonant delocalization for random Schrödinger operators on tree graphs" http://arxiv.org/abs/1104.0969.

3. M. Aizenman, S. Warzel, " Absence of Mobility Edge for the Anderson Random Potential on Tree Graphs at Weak Disorder, (in preparation, to be submitted).