The Hebrew University of Jerusalem האוניברסיטה העברית בירושלים
מכון רקח לפיסיקה
The Racah Institute of Physics

Mode-locked laser pulse fluctuations

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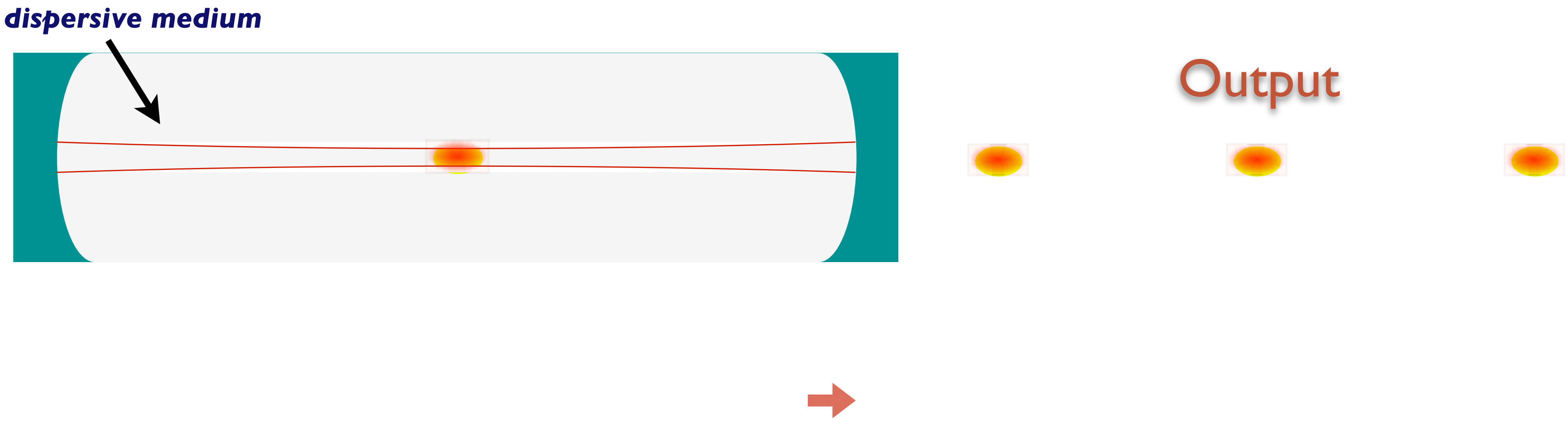
Joint work with: Michael Katz, Baruch Fischer

Acknowledgement: Rafi Weill, Oded Basis, Alex Bekker, Vladimir Smulakovsky
Supported by: Israel Science Foundation

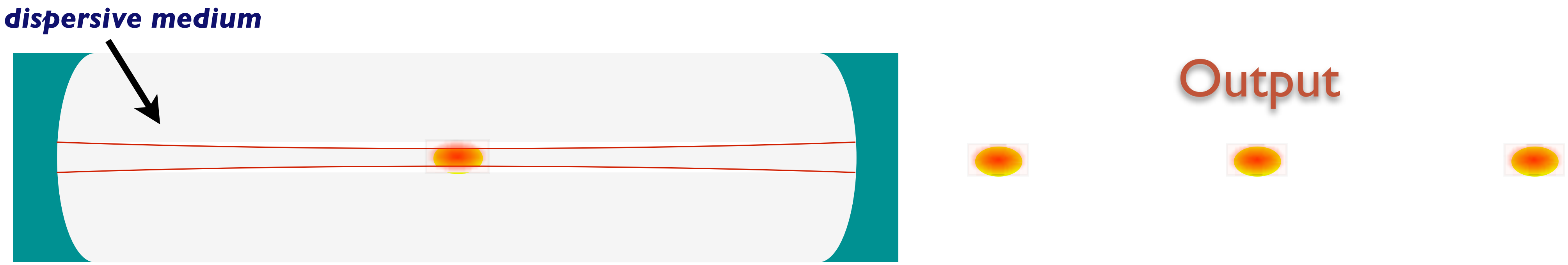
Synopsis

- Mode-locked soliton lasers and noise
- The statistical steady state
- Fluctuations in the steady state
    1. Pulse-continuum interactions
    2. Gain fluctuations
    3. Slow modes of pulse dynamics
    4. Pulse parameter equations of motion
- Autocorrelation and diffusion of pulse parameters

Mode-locked soliton lasers

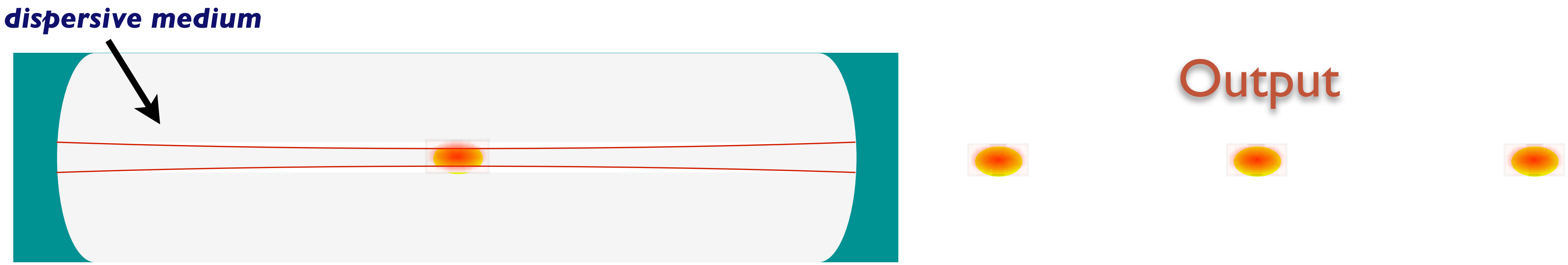


Mode-locked soliton lasers



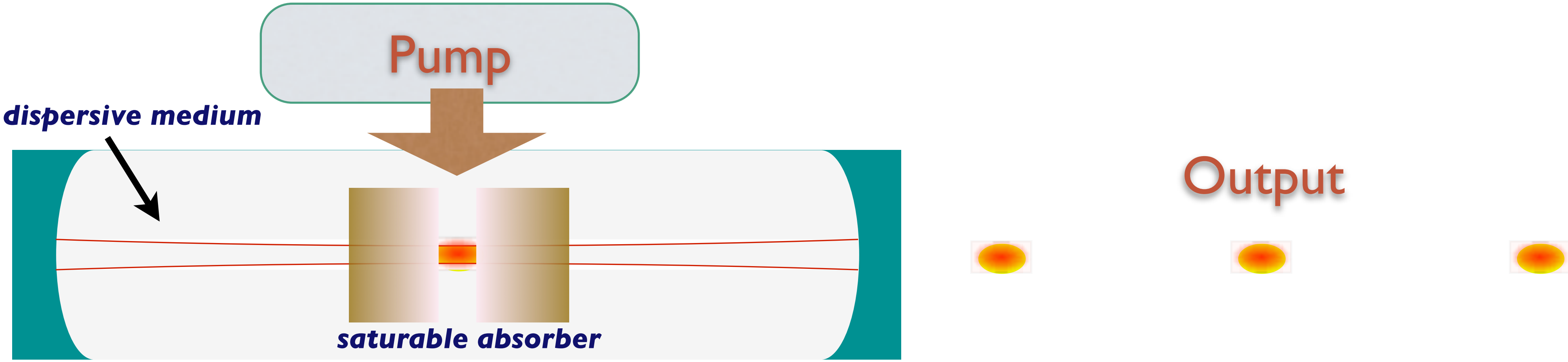
- Ultrashort light pulses <1ps ➡ high intensity, broad bandwidth

Mode-locked soliton lasers



- Ultrashort light pulses <1ps ➡ high intensity, broad bandwidth
- Dominant dispersive effects:
  - Chromatic dispersion ("anomalous") — *linear*
  - Kerr effect — *nonlinear*

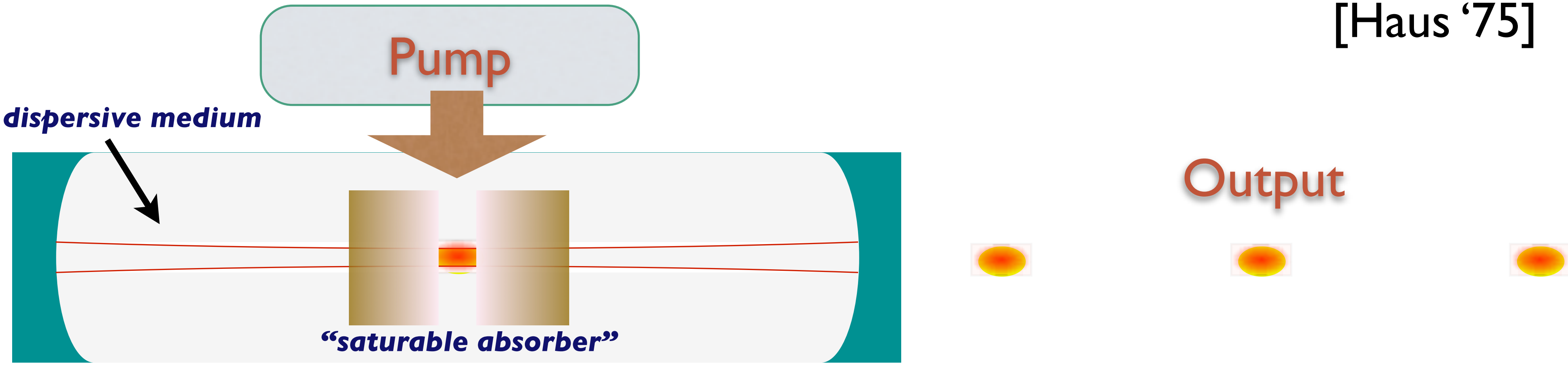
Mode-locked soliton lasers



- Pulse shaping effects:
  - Overall gain & gain filtering — *linear*
  - "Saturable absorption": Intensity-bleached absorbing element — *nonlinear*
- Relatively weak: proportional to $\mu \ll 1$

Mode-locked soliton lasers

[Haus '75]

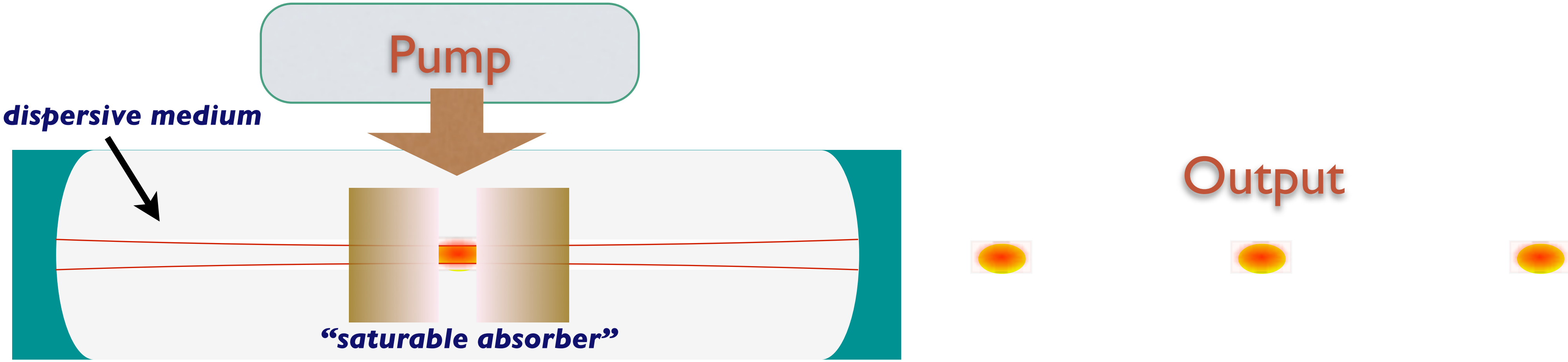


- "Master" equation of motion for the field envelope ψ

$$\boxed{\partial_z \psi = (i+\mu)\left(\frac{1}{2}\partial_t^2 \psi + |\psi|^2 \psi\right) + g\psi}$$

dispersive effects (*non-dimensionalized*) pulse shaping overall net gain (<0)

Mode-locked soliton lasers



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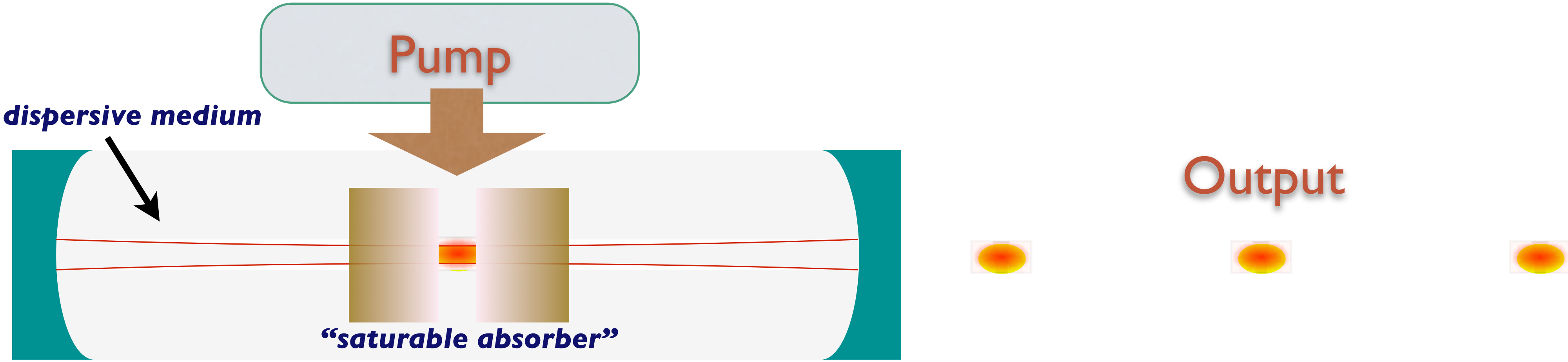
- Soliton-like pulse: $\boxed{\psi_s(t,z) = a\,\mathrm{sech}\left(\frac{t-C(z)}{\tau}\right)e^{i\Phi(t,z)}}$

- Parameters: $a = \frac{1}{2}P$ $\qquad C = c - \int V dz$

$$\tau = \frac{1}{a} \qquad \Phi = \phi + \frac{1}{4}Vt + \frac{1}{8}\int (P^2+V^2)dz$$

Mode-locked soliton lasers



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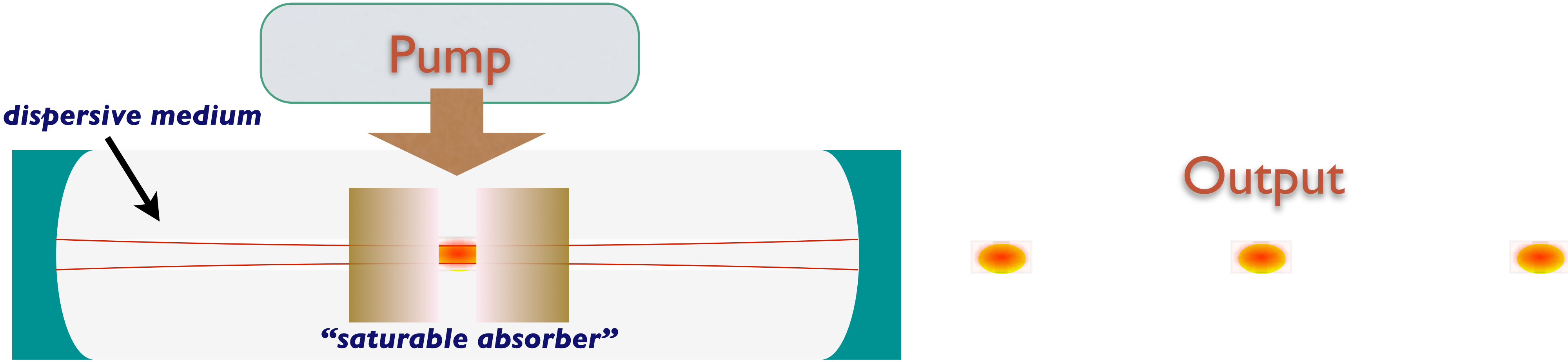
- Timing
- Phase

- Power
- Frequency

Mode-locked soliton lasers



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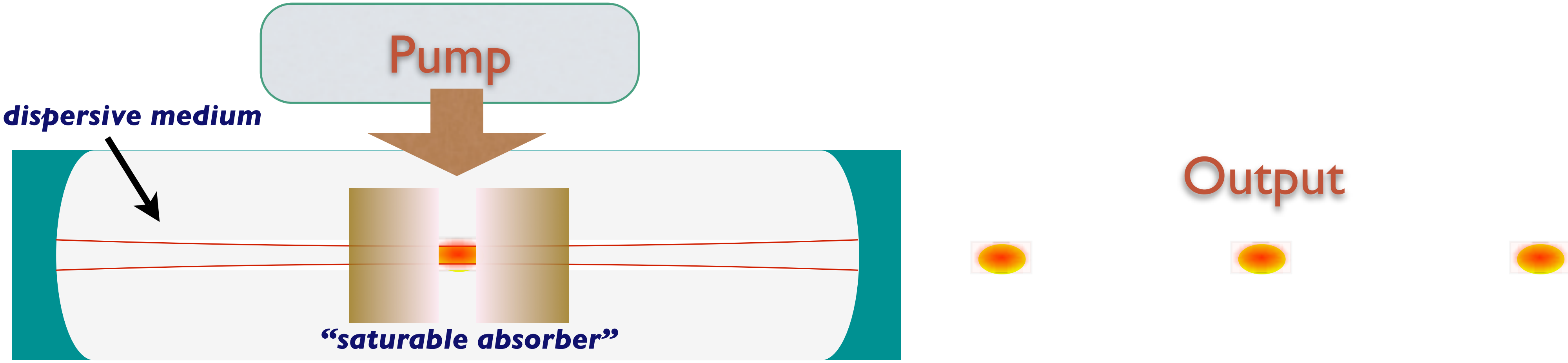

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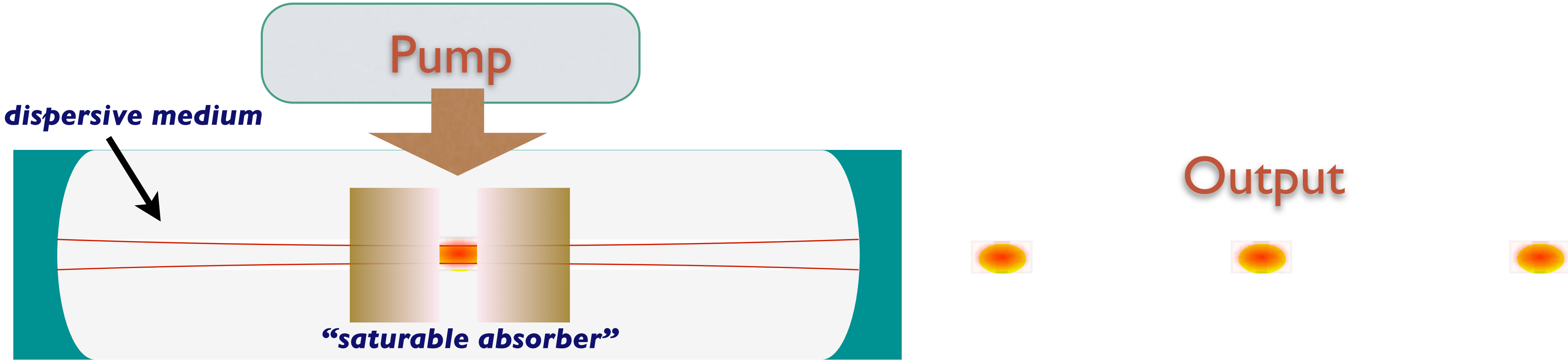
Mode-locked soliton lasers



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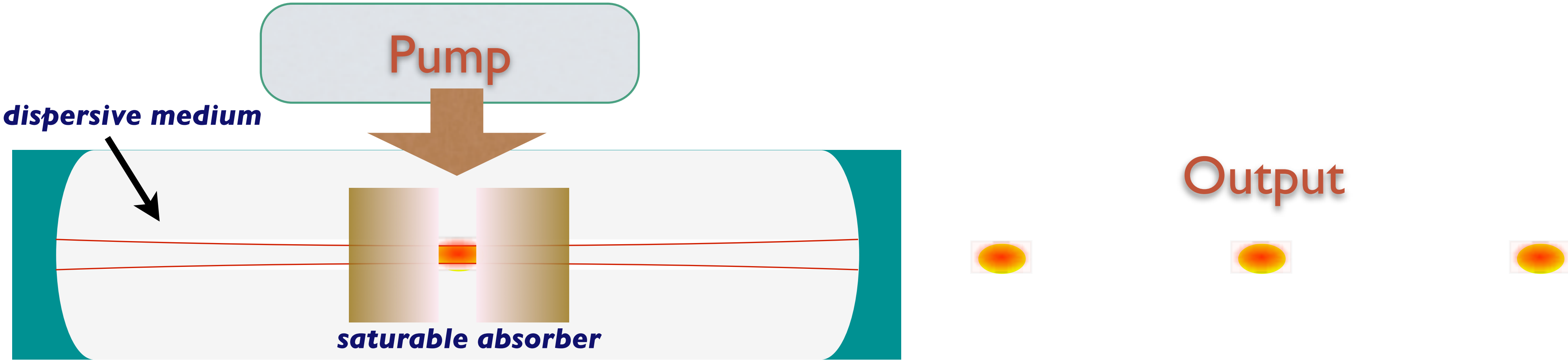

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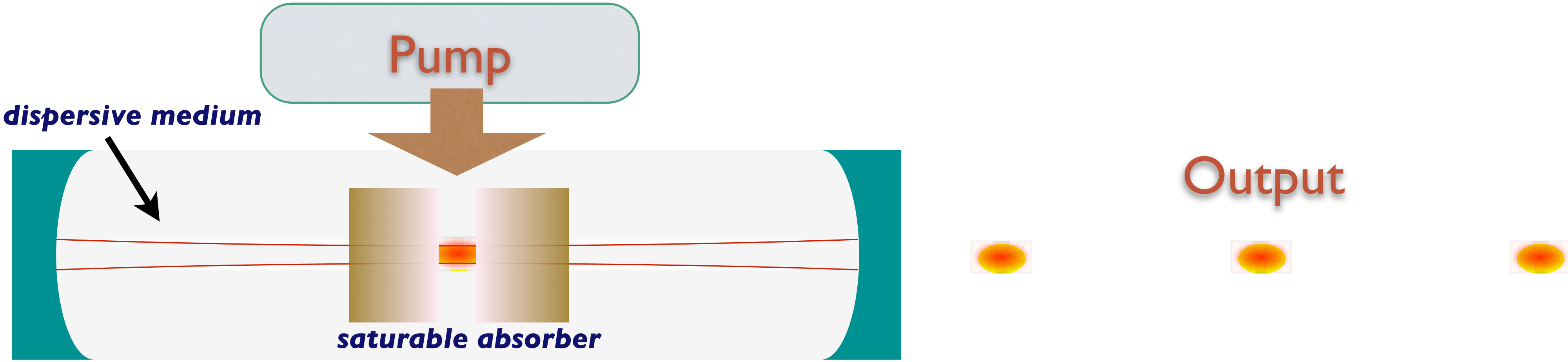
- Pulse parameters:
  - Power $\boxed{P = \sqrt{\frac{8|g|}{\mu}}}$ & frequency $\boxed{V = 0}$ fixed
  - Pulse shaping: Singular perturbation
  - Timing c and phase ϕ free — exact symmetries

Mode-locked soliton lasers



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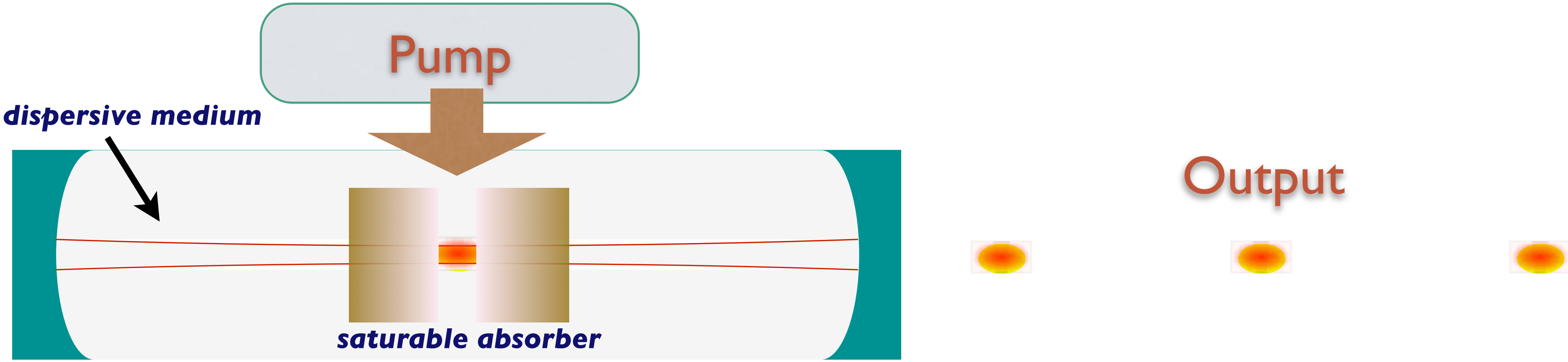

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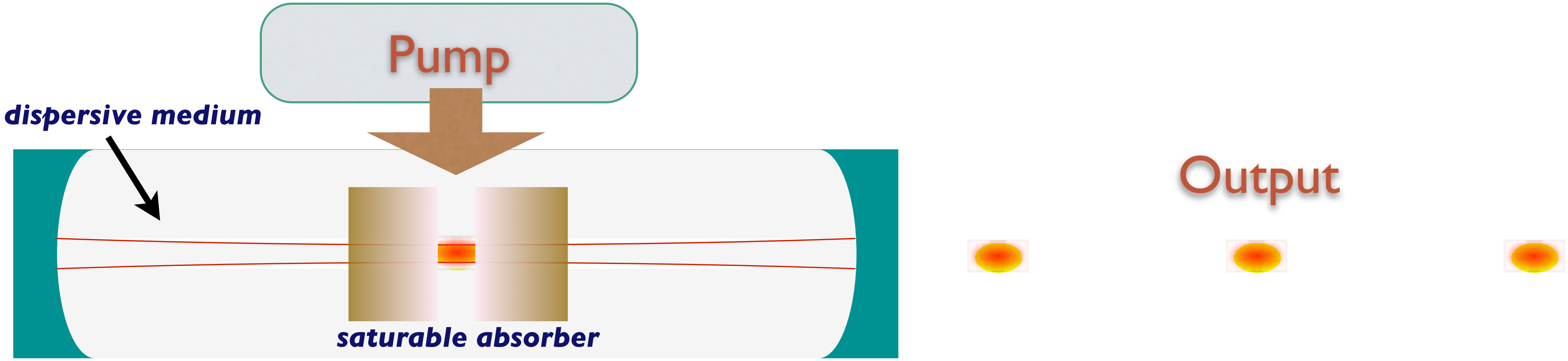

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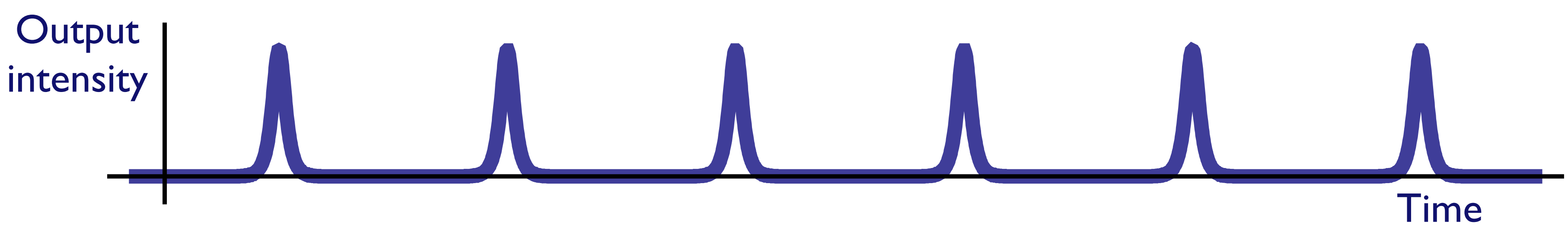
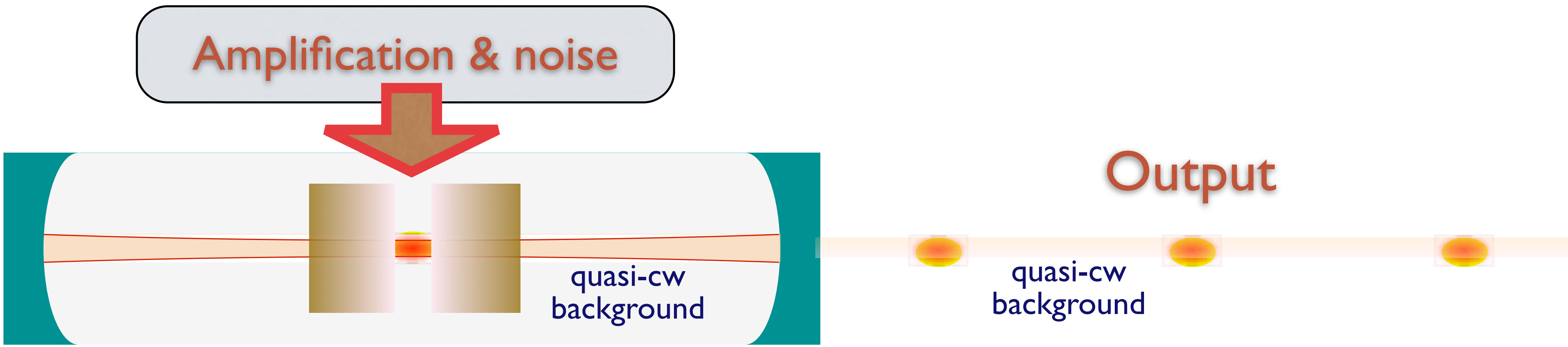
- Ideal output: Periodic pulse train



Noise and fluctuations



- Noisy master equation:

$$\partial_z \psi = (i+\mu)\left(\frac{1}{2}\partial_t^2\psi + |\psi|^2\psi\right) + g\psi \boxed{+\,\epsilon\Gamma(z,t)}$$

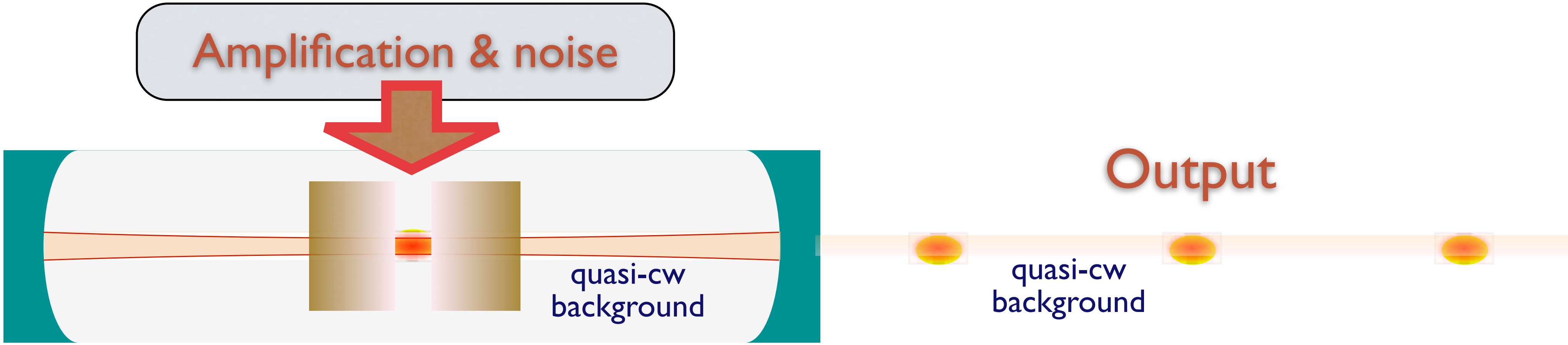
$\epsilon \ll 1$

- Weak Gaussian white noise

$$\langle \epsilon\Gamma(z_1,t_1)\epsilon\Gamma^*(z_2,t_2)\rangle = 2\epsilon^2 T L\delta(z_1-z_2)\delta(t_1-t_2)$$

Noise power
injection rate

Noise and fluctuations



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- Weak Gaussian white noise

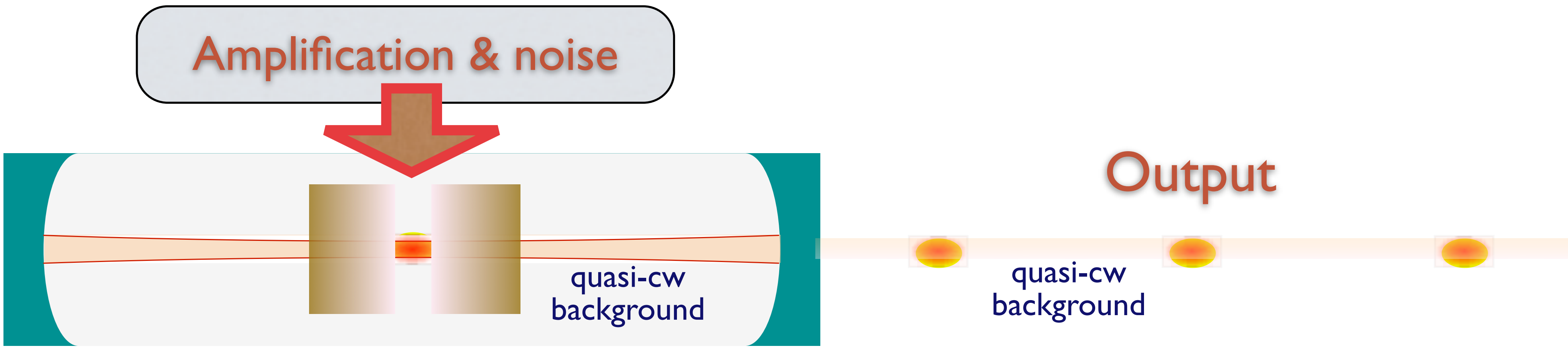
$$\langle \epsilon\Gamma(z_1,t_1)\epsilon\Gamma^*(z_2,t_2)\rangle = 2\boxed{\epsilon^2 T} L\delta(z_1 - z_2)\delta(t_1 - t_2)$$

Noise power injection rate

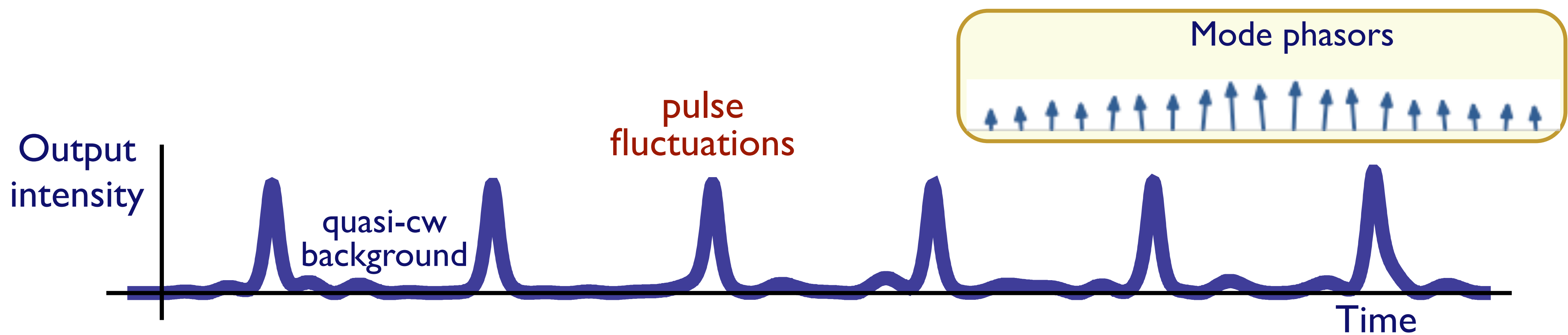
- Waveform is perturbed

$$\boxed{\psi = \psi_s(t,z) + O(\epsilon)}$$

Statistical light-mode dynamics



- Perturbed waveform: $\psi = \underbrace{\psi_s(t,z)}_{\text{Ordered pulse waveform}} + \underbrace{O(\epsilon)}_{\text{Noisy waveform}}$



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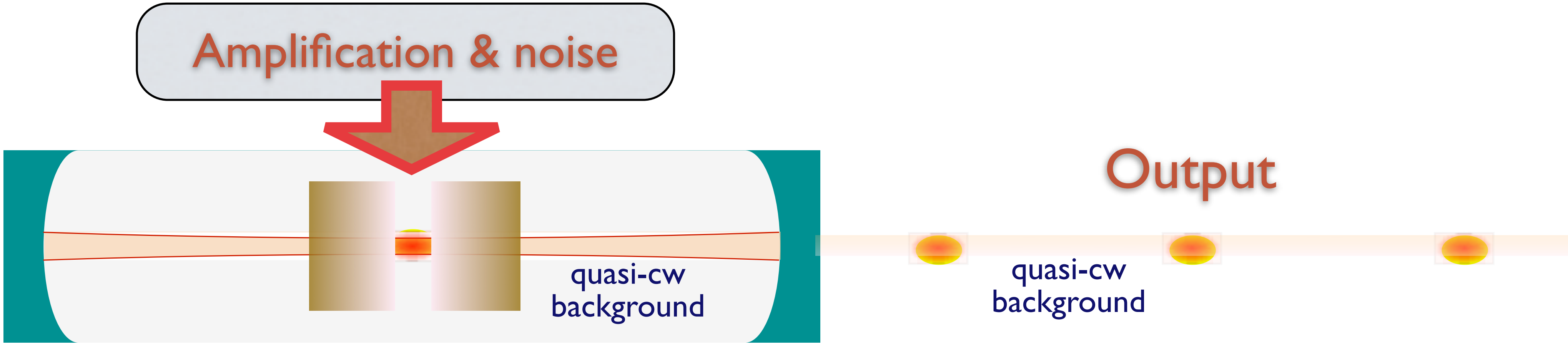

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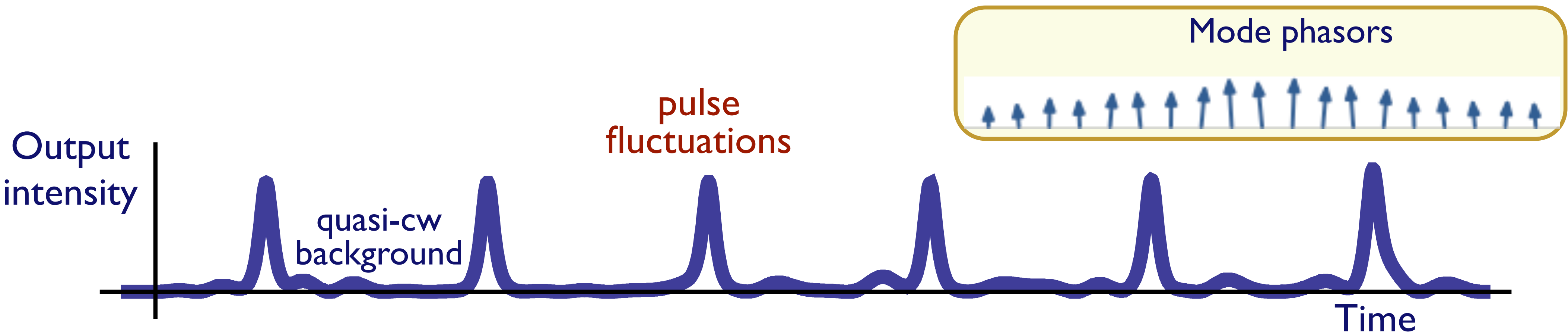
  1. Pulse fluctuations



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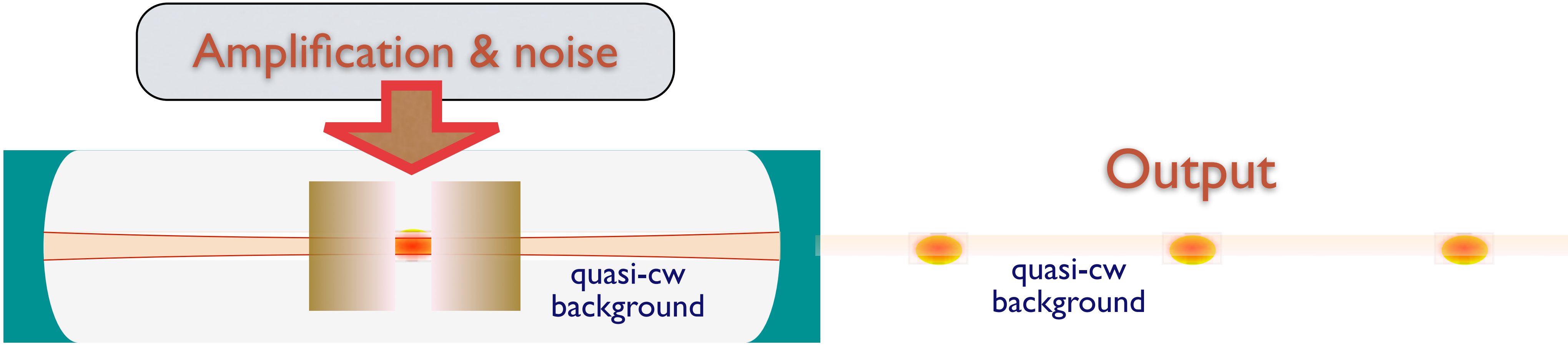
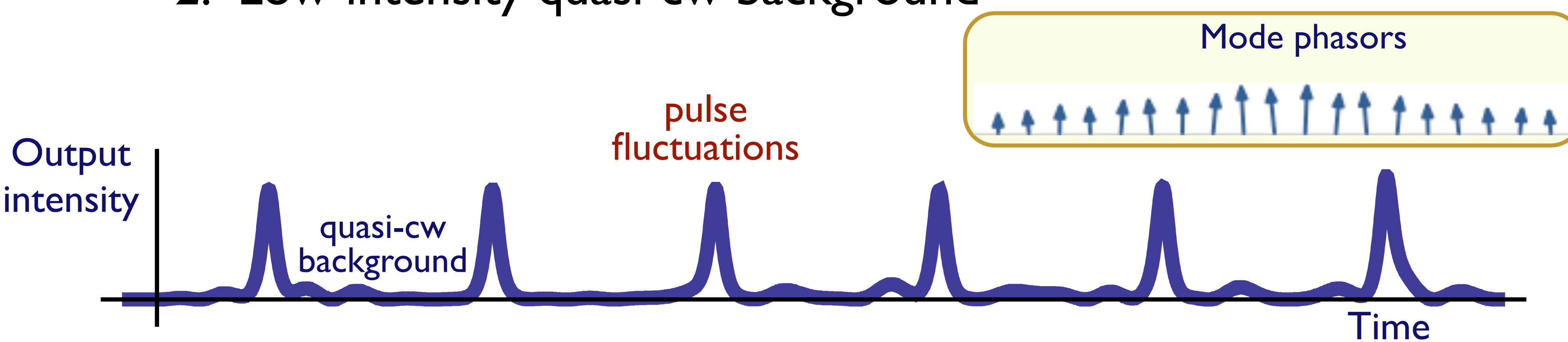


- Perturbed waveform: $\psi = \psi_s(t,z) + O(\epsilon)$

  Ordered pulse waveform   Noisy waveform

  1. Pulse fluctuations
  2. Low-intensity quasi-cw background



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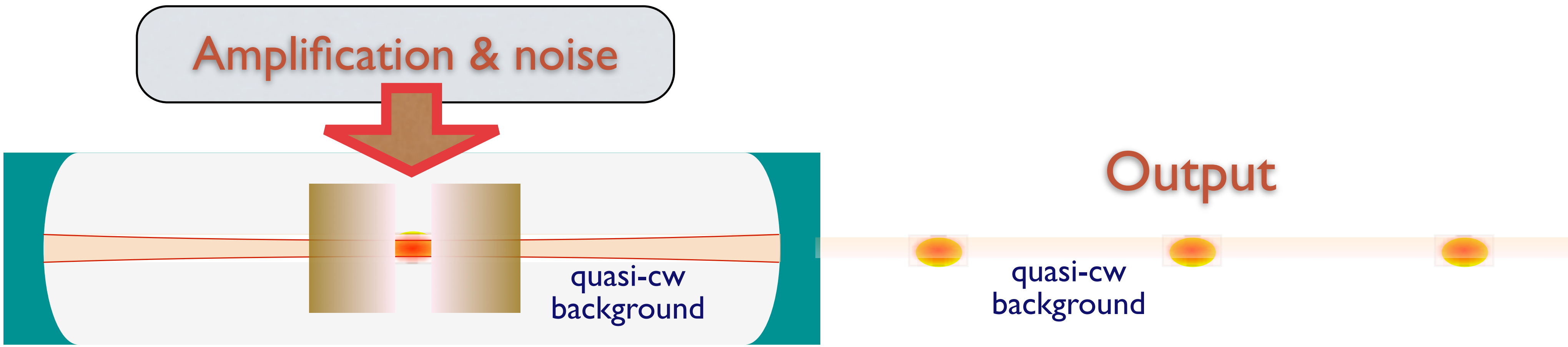
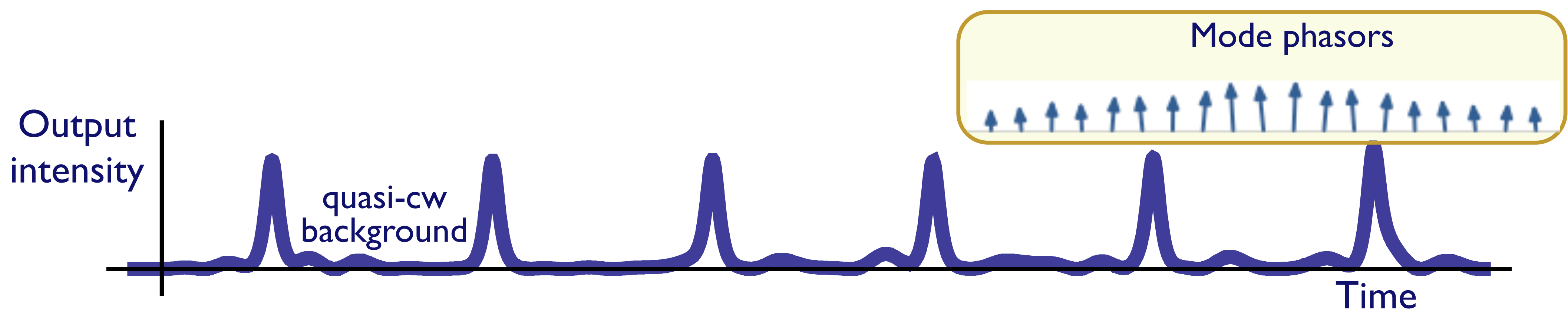


- Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow
- Continuum waveform $\epsilon\psi_c$: Weak & wide



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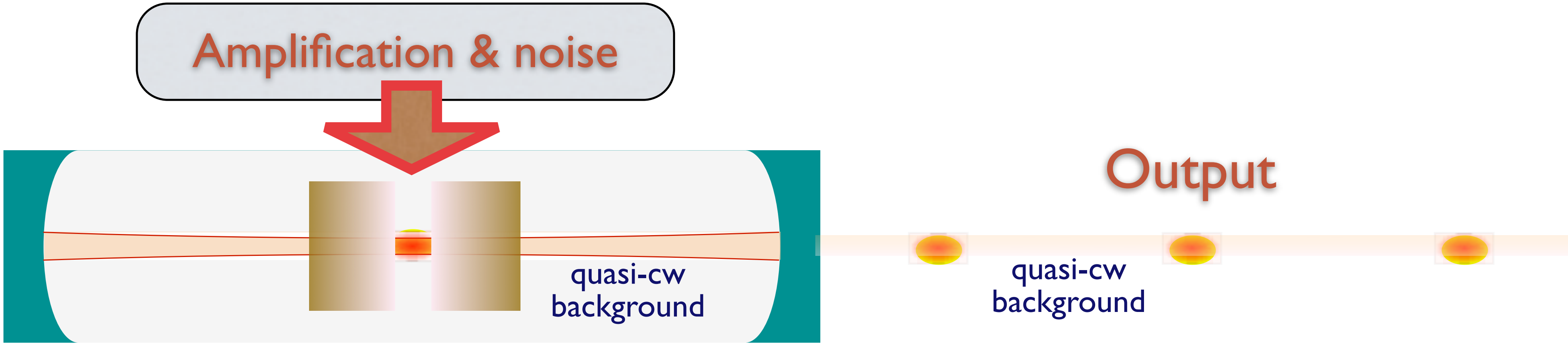

- Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow
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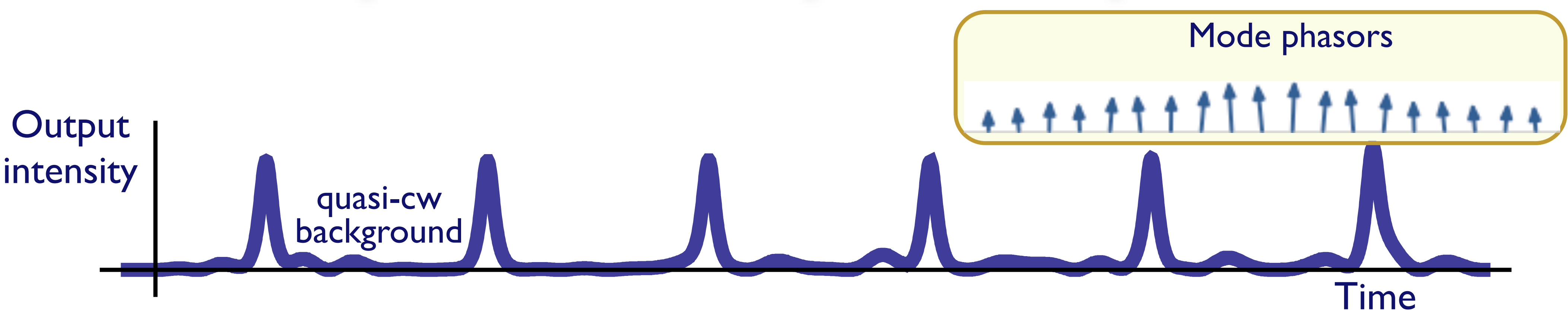
    - Pulse power ~ Continuum power ~ total power



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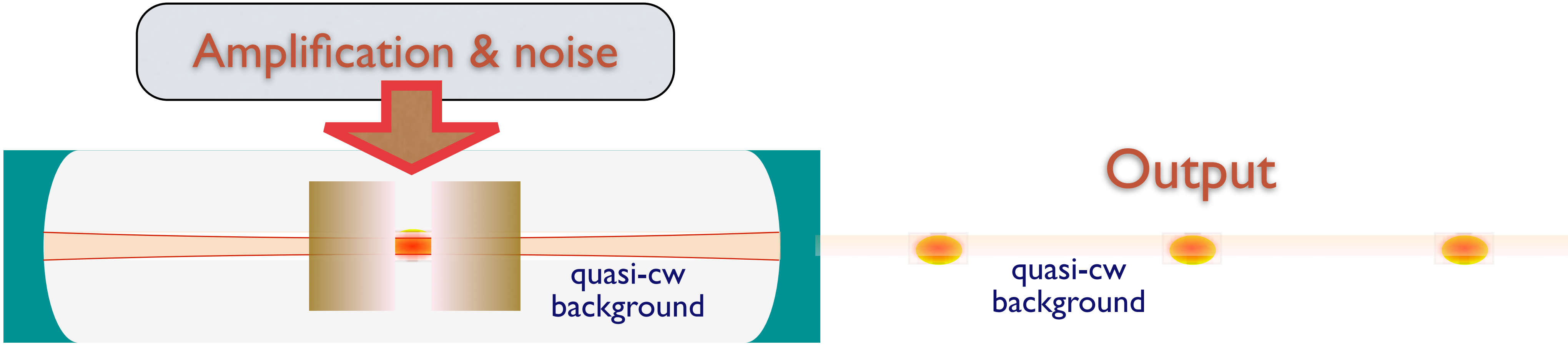
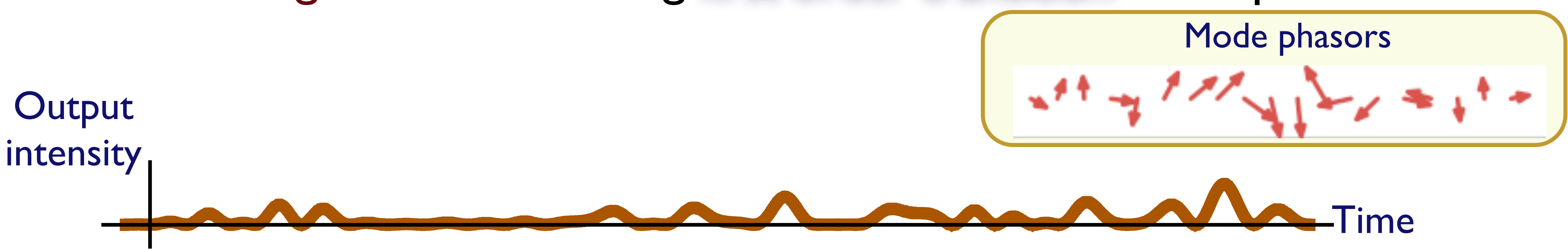
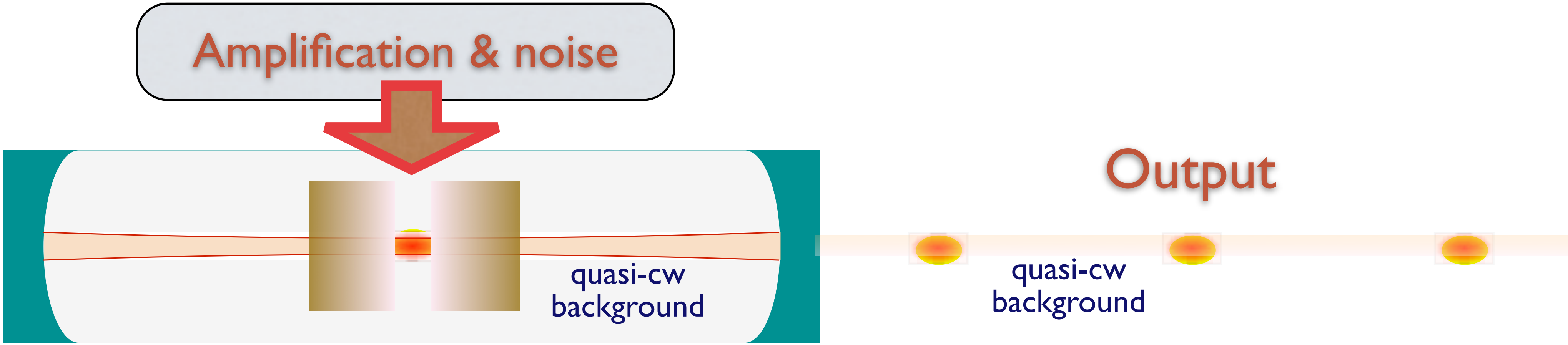


- ~~Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow~~
- Continuum waveform $\epsilon\psi_c$: Weak & wide
- ~~Pulse power~~ ~ Continuum power ~ total power
- For strong noise, disordering first order transition to cw phase



SLD: The gain balance method



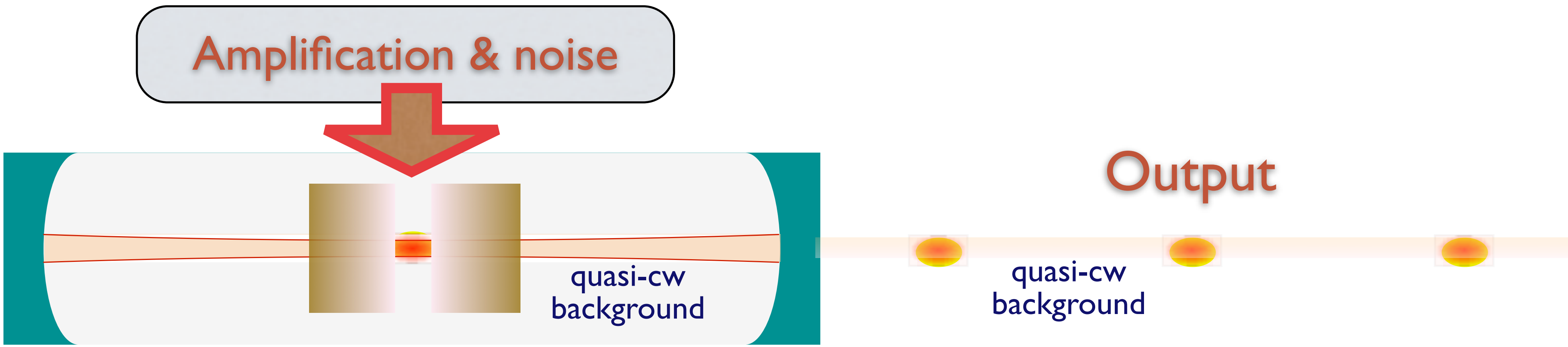
$\psi_s \cancel{+ \epsilon\psi_1}$

$\epsilon\psi_c$

P

$\mathcal{P}$

SLD: The gain balance method



- Pulse waveform $\psi_s \cancel{+ \epsilon\psi_1}$: Strong & narrow → Neglect noise
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- Pulse power P + continuum power = total power $\mathcal{P}$

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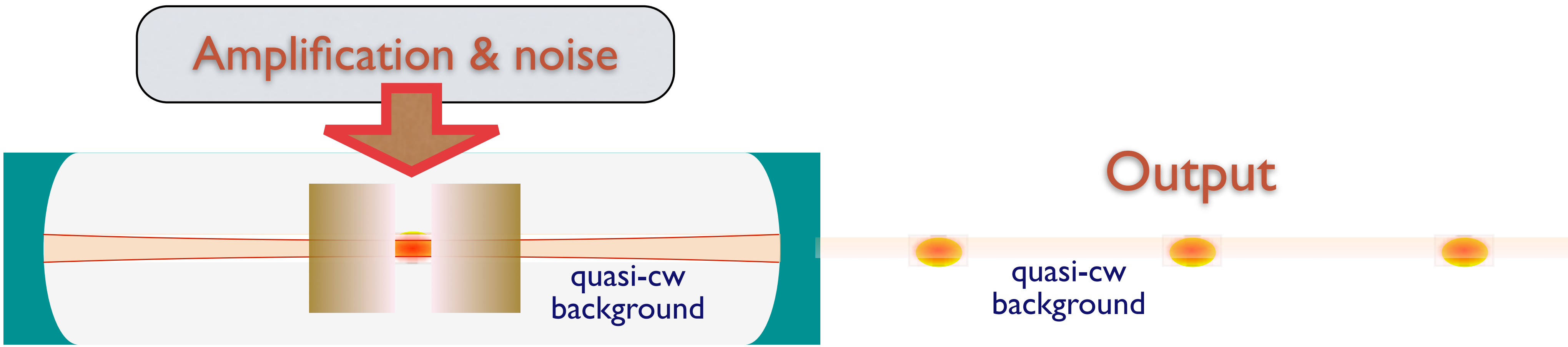

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Net gain g

- Common gain value determines the power distribution between pulse & continuum

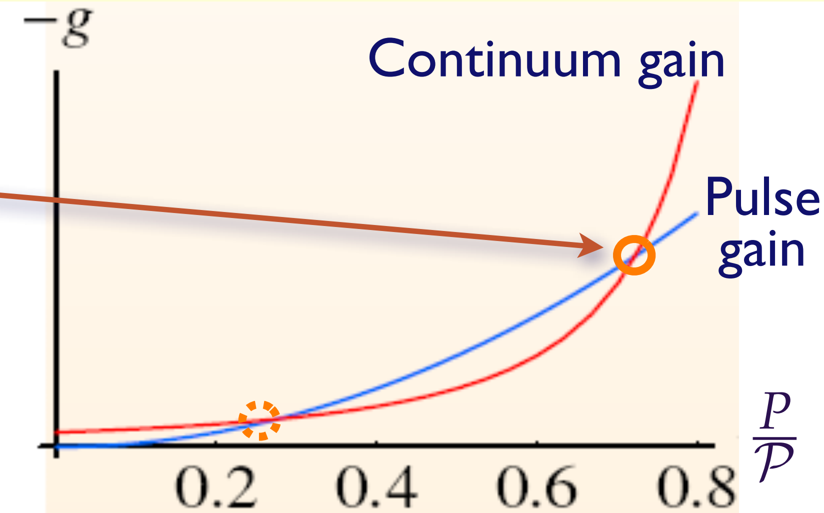
SLD: The gain balance method

- Steady state power distribution

$$\frac{P}{\mathcal{P}} = \frac{1}{2}\left(1+\sqrt{1-\frac{\epsilon^2L^2}{\mu}\frac{8T}{\mathcal{P}^2}}\right)$$

- Thermodynamic limit $L \gg 1$

$$\frac{\epsilon}{\sqrt{\mu}} \ll 1$$



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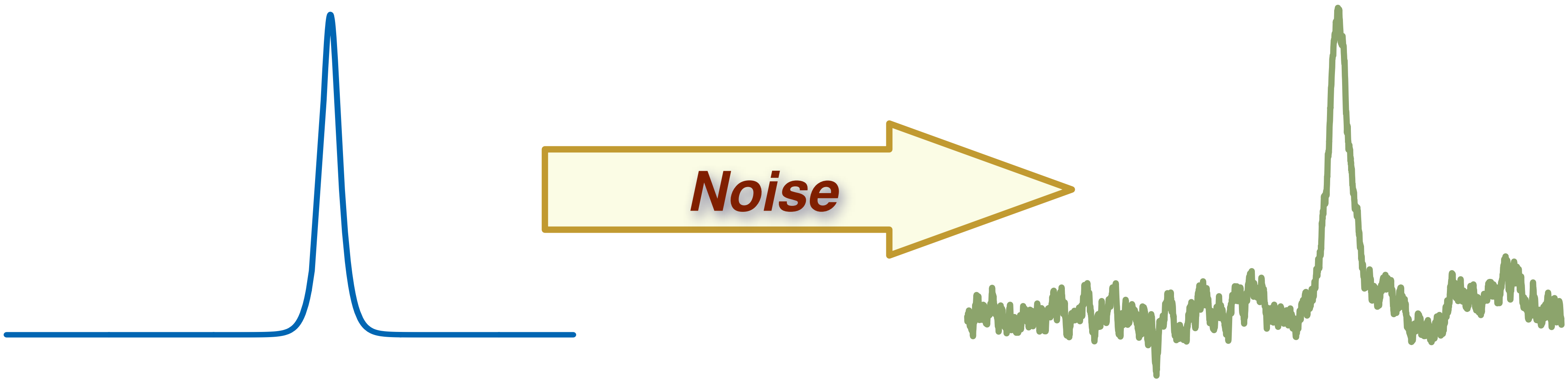
Net gain g

- Mode locking is possible *only* if a consistent power distribution between pulse & continuum exists

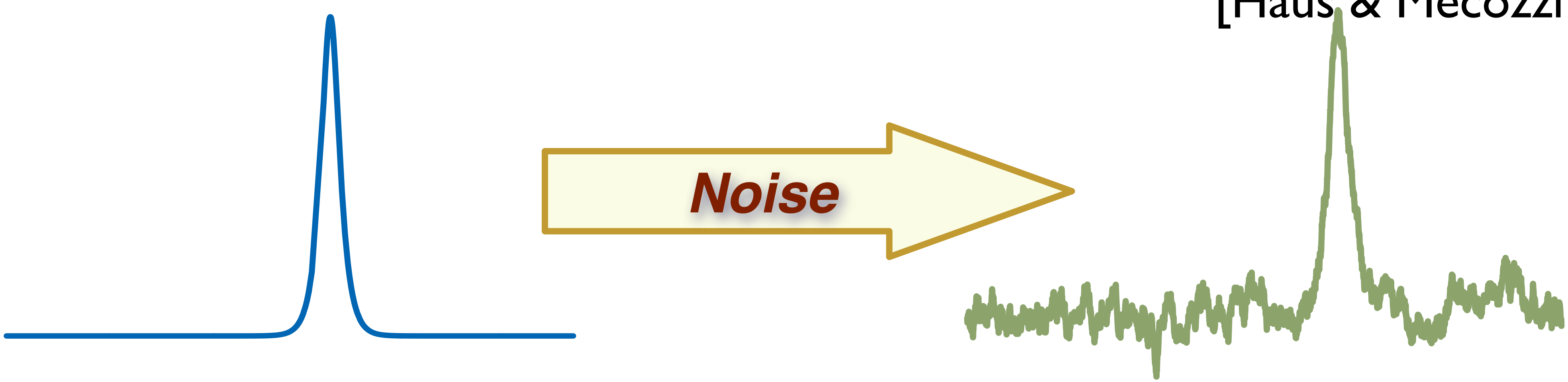
Pulse fluctuations



- Noise causes jitter in pulse parametrs
  - Power P & frequency V fluctuate
  - Timing c and phase ϕ diffuse

Pulse fluctuations

[Haus & Mecozzi '93]



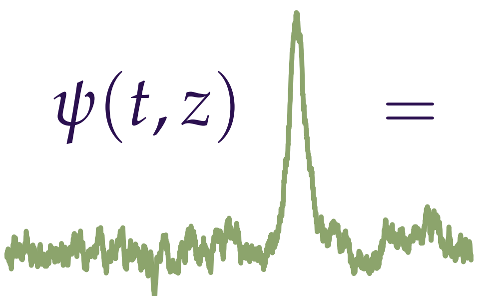
- Question: *What are the statistical properties of the fluctuations?*
- Practical implications:
    - Performance of pulse sources
    - Precision of frequency-comb metrology

SLD beyond thermodynamics

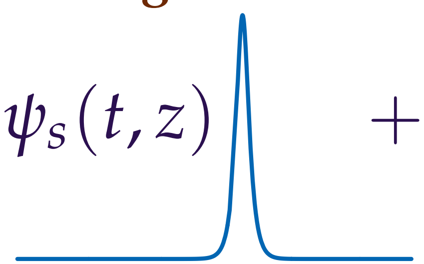
- Idea: Decompose wave form in 3 parts

strong & narrow | weak & *wide* | weak & narrow

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$

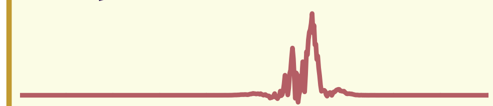


fluctuating waveform, power = $\mathcal{P}$



soliton-like pulse with fluctuating parameter $c(z),\ \phi(z),\ P(z),\ V(z)$

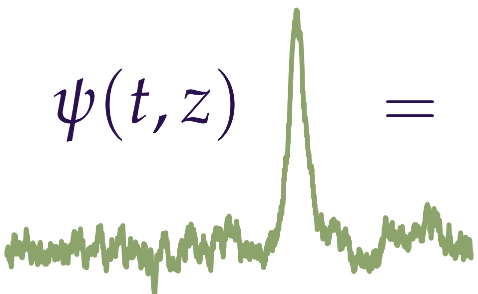
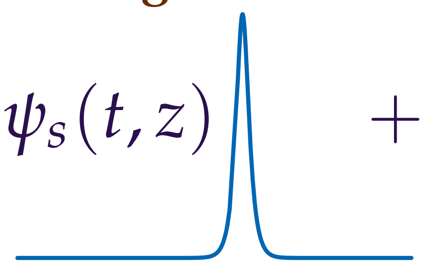
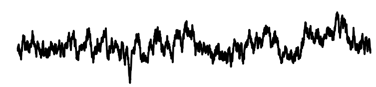
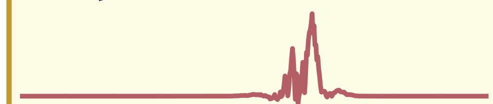
linear continuum, power ~ $\mathcal{P} - P(z)$



overlap waveform power $\ll \mathbf{1}$

SLD beyond thermodynamics

- Idea: Decompose wave form in 3 parts

	strong & narrow	weak & *wide*	weak & narrow
$\psi(t,z) =$	$\psi_s(t,z) +$	$\epsilon\psi_c(t,z) +$	$\epsilon\psi_1(t,z)$
			
fluctuating waveform, power = $\mathcal{P}$	soliton-like pulse with fluctuating parameter	linear continuum, power ~ $\mathcal{P} - P(z)$	**overlap** waveform power $\ll 1$

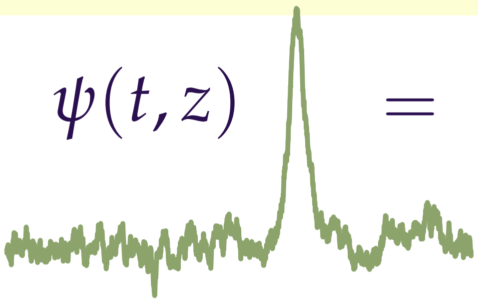
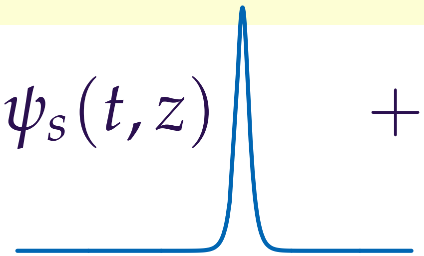
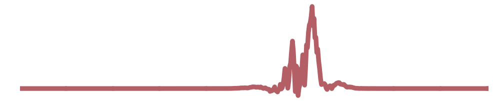
$c(z),\ \phi(z),\ P(z),\ V(z)$

- Negligible in steady-state
- Important for fluctuations

- Main goal: Calculate statistics of pulse paramaters

SLD beyond thermodynamics

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$

$c(z),\ \phi(z),\ P(z),\ V(z)$

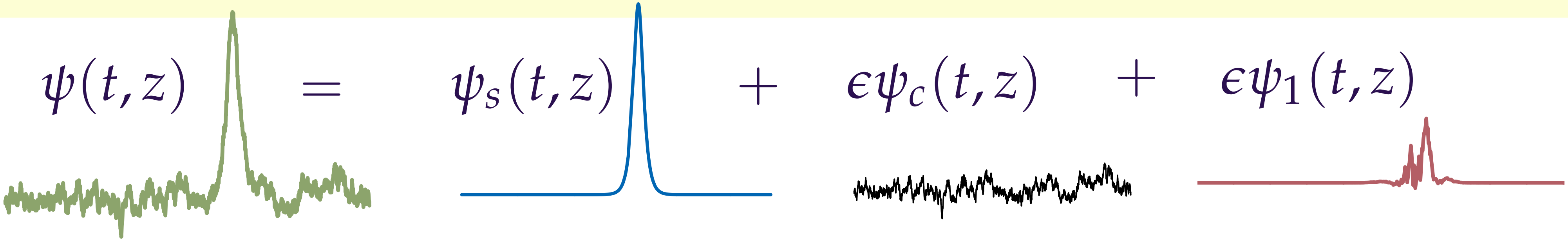
- New qualitative effects:

1. **Oscillations** in power & frequency correlation functions

2. **Enhancement** of phase diffusion rate

1. The perturbation equation

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



- Let $P = P_0 + \frac{\epsilon}{\sqrt{\mu}}p(z),\ V = \frac{\epsilon}{\sqrt{\mu}}v(z),\ c \rightarrow \frac{\epsilon}{\sqrt{\mu}}c(z),\ \phi \rightarrow \frac{\epsilon}{\sqrt{\mu}}\phi(z)$

steady-state pulse power

fluctuations small parameter

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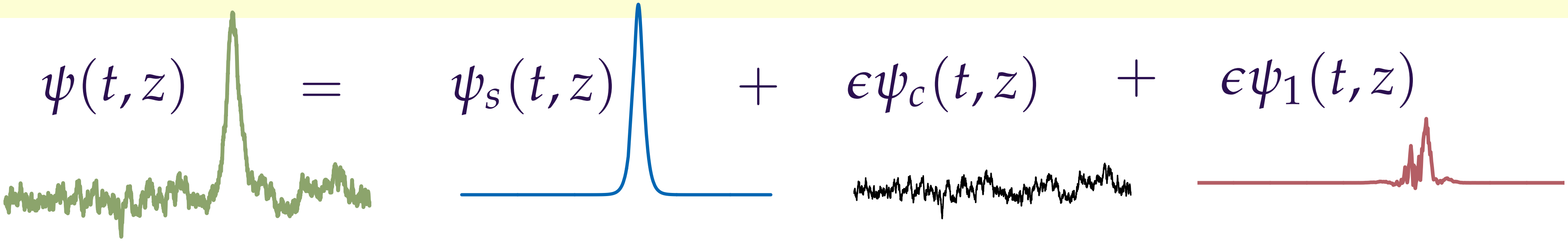

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- Linearized master equation for ψ_1

$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}} \partial_z \vec{x} \cdot \nabla_x \psi_s = L\psi_1 - i\tfrac{1}{2}\sqrt{\mu}\, v(z)\partial_t \psi_s + g_1 \psi_s + f$$

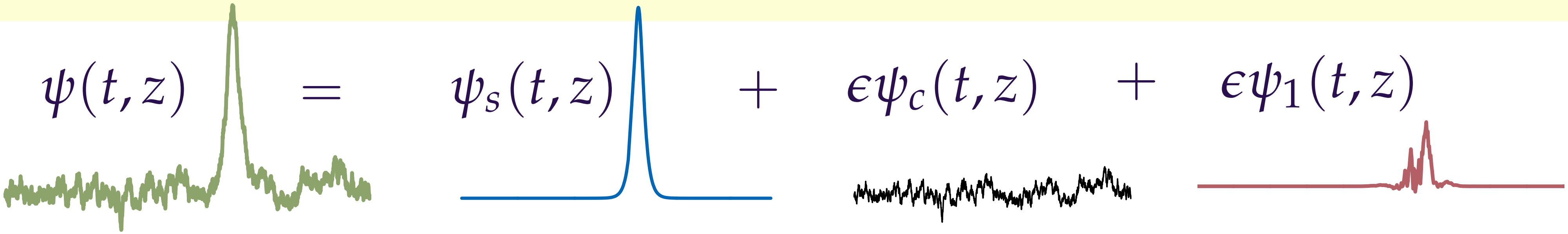
parameter set

gain fluctuations

- Nonlinearity-generated forcing by overlap of ψ_s & ψ_c

2. Gain fluctuations

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



- Double role of net gain in the steady state:

1. Determines the pulse & continuum power $\boxed{g_0 = -\frac{\mu}{8}P_0}$

   Gain balance

2. Determined by the overall power $\boxed{g_0 = g(\mathcal{P})}$

   Gain saturation

2. Gain fluctuations

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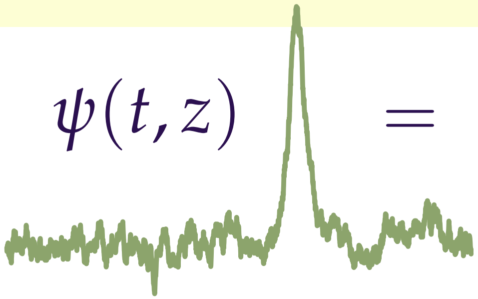
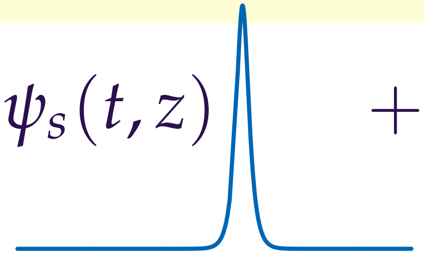

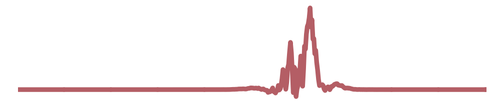

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- Gain fluctuations $g = g_0 + \epsilon g_1$ arise from

1. Random shuffle of power between pulse and continuum
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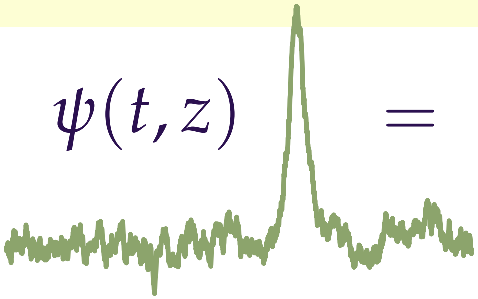
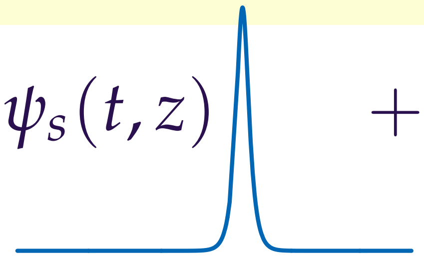

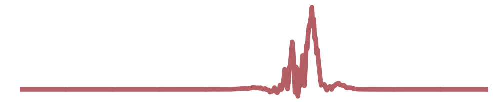

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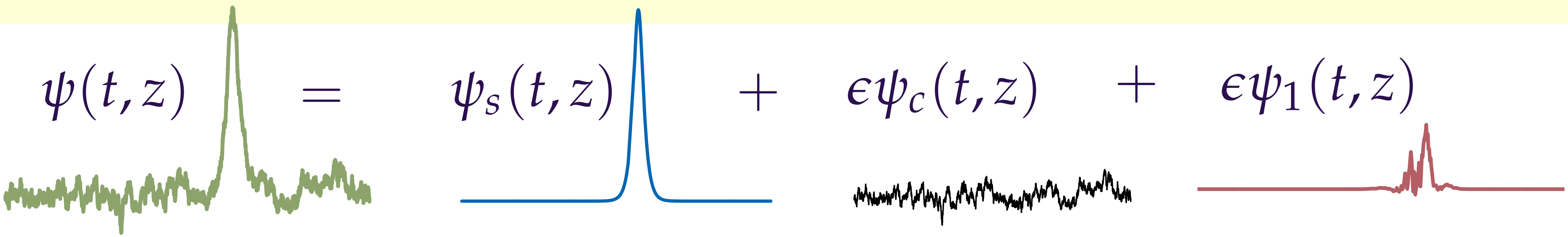

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- Linearized master equation for ψ_1

$$\partial_z\psi_1 + \frac{1}{\sqrt{\mu}}\partial_z\vec{x}\cdot\nabla_x\psi_s = L\psi_1 - i\tfrac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + g_1\psi_s + f$$

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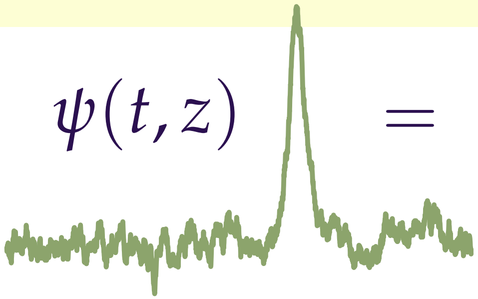
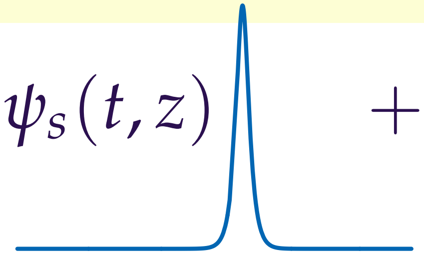

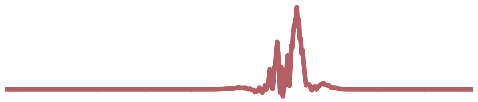

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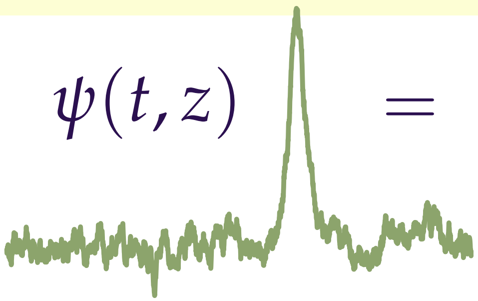
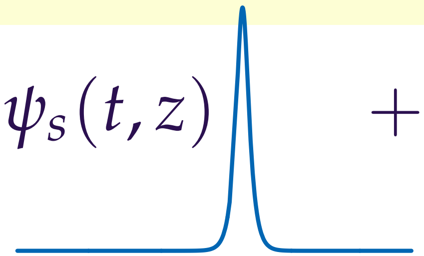
- Fluctuation conserve overall power

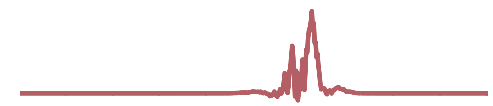
$$\boxed{g_1\mathcal{P} = -\frac{\sqrt{\mu}P_0^2}{4}p - \int dz\psi_s^*\left(\Gamma + L(\psi_c + \psi_1)\right)}$$

3. The slow modes

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$







- Perturbation equation including gain fluctuations:

$$\partial_z\psi_1 + \frac{1}{\sqrt{\mu}}\partial_z\vec{x}\cdot\nabla_x\psi_s = \tilde{L}\psi_1 - i\frac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + \tilde{f}$$

Linear operator including rank-1 gain fluctuations term

Forcing including gain fluctuations

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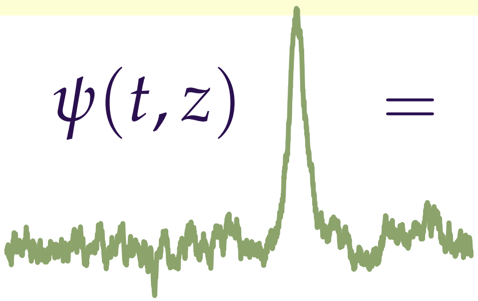
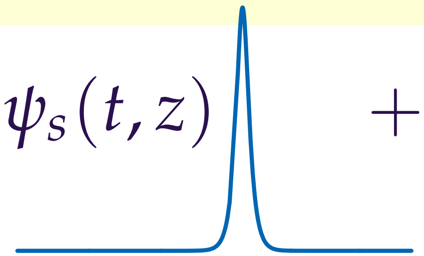

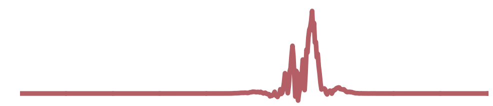

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Linear operator including rank-1 gain fluctuations term

Forcing including gain fluctuations

- Remaining arbitrariness: A slight shift in $\vec{x}$ can be absorbed in ψ_1, $\psi_s(\vec{x}) + \epsilon\psi_1 = \psi_s(\vec{x}+\delta\vec{x}) + \epsilon(\psi_1 + \delta\psi_1)$

3. The slow modes

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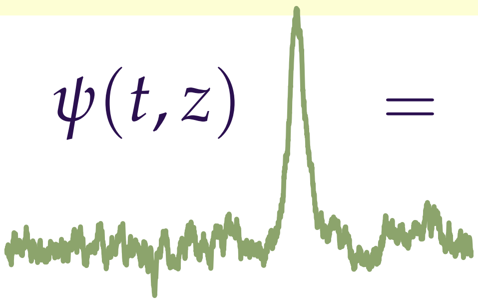
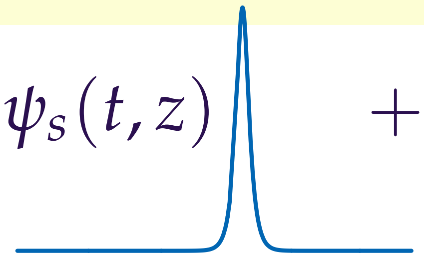

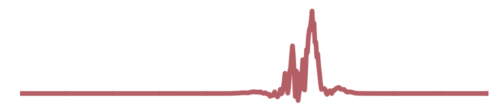

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- *Q*: How to define the pulse parameters?
- *A*: Let ψ_1 lie outside 4-dimensional slow eigen-space of $\tilde{L}$

3. The slow modes

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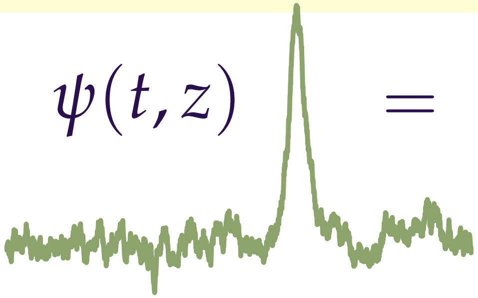 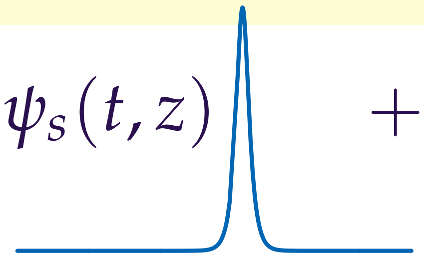  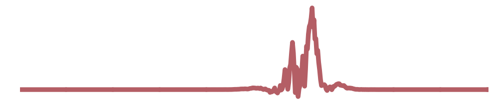

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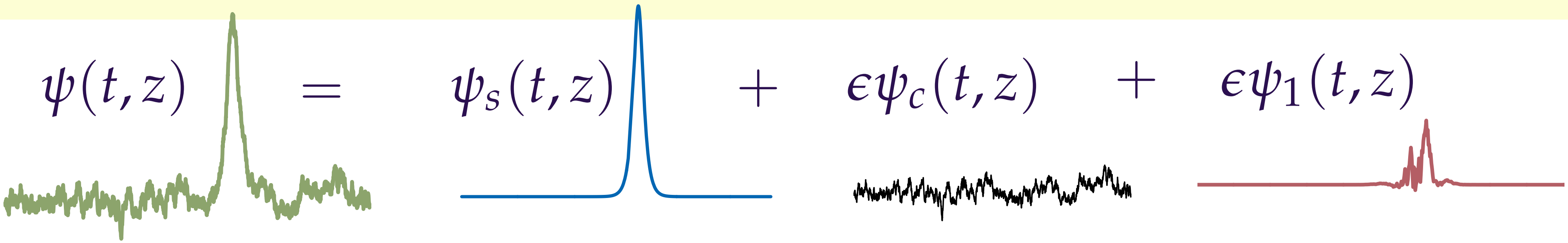
- *Recall:* linearized NLS has 4-d zero eigen-space:

timing, phase:
annihilated by L_{NLS}

frequency, power:
annihilated by L^2_{NLS}

3. The slow modes



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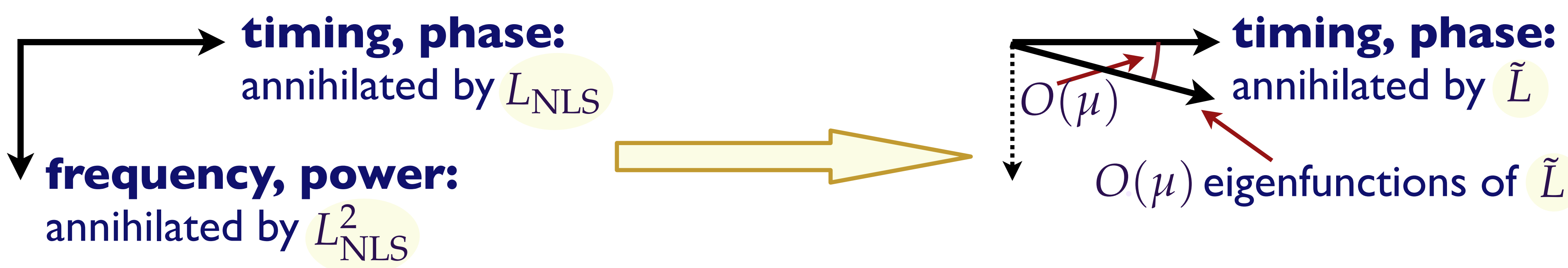
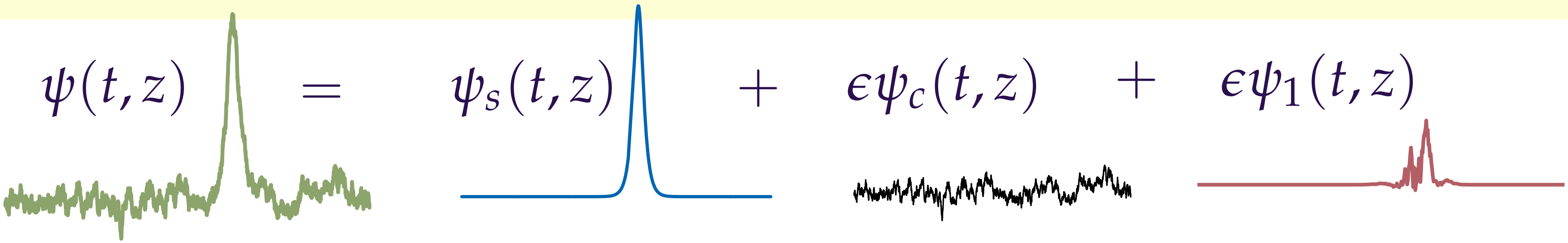


- Degeneracy is half-lifted by $\tilde{L} = L_{\mathrm{NLS}} + O(\mu)$

4. Pulse parameter dynamics

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$



- Perturbation equation:

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- *where:* ψ_1 lies outside 4-dimensional slow eigen-space of $\tilde{L}$

$\langle q_n, \psi_1\rangle = 0$ where $q_1,\ldots,q_4$ are slow *left* $\tilde{L}$ eigenfunctions

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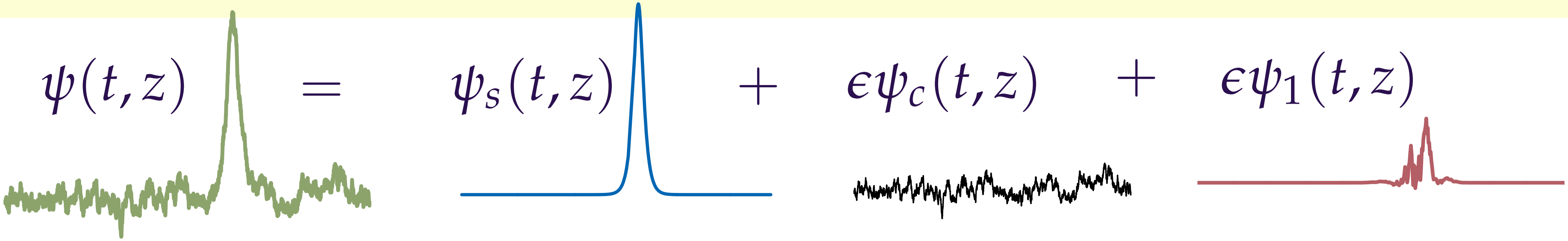

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- $\langle q_n, \cdot \rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion

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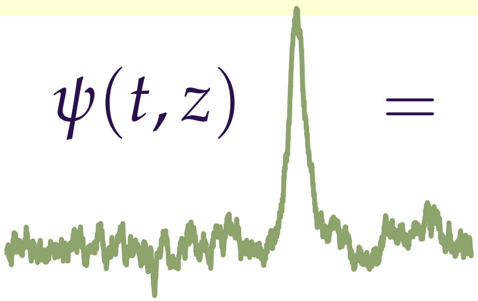
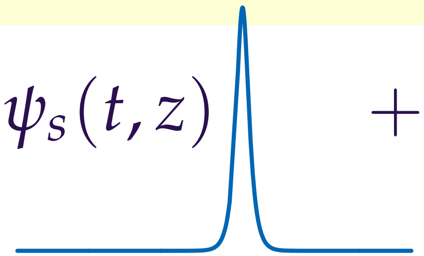

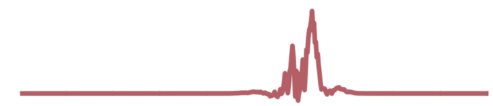

- $\langle q_n, \cdot \rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion:

Power:	$\partial_z p = -\mu \frac{P_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p$
Phase:	$\partial_z \phi = \sqrt{\mu} f_\phi$
Frequency:	$\partial_z v = -\mu \frac{P_0^2}{6} v - \sqrt{\mu} f_v$
Timing:	$\partial_z c = \sqrt{\mu} f_c$

Restoring terms

Random forcing

4. Pulse parameter dynamics

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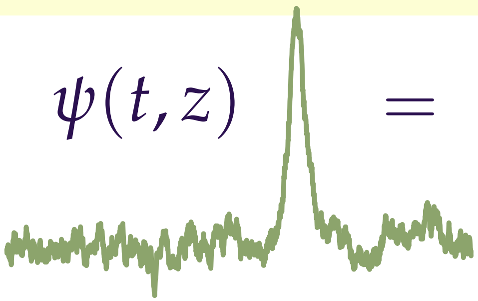
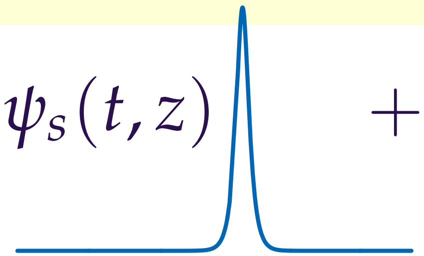

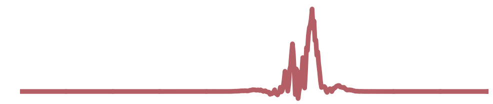

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Bounded fluctuations

Restoring terms

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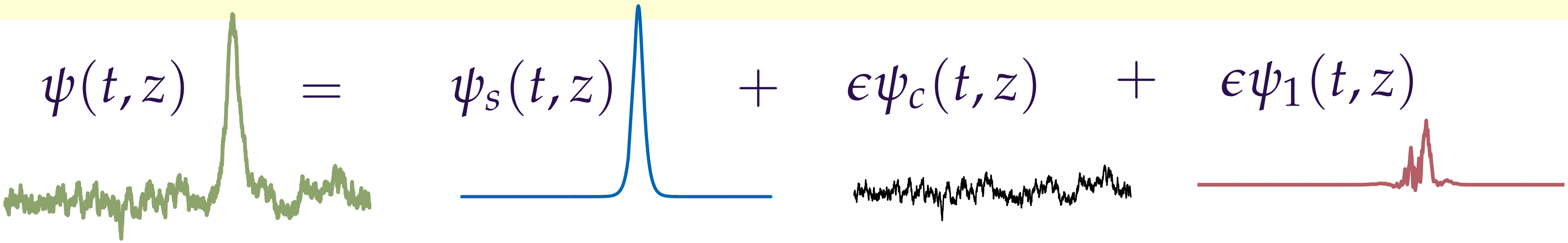

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Bounded fluctuations

Diffusion

Restoring terms

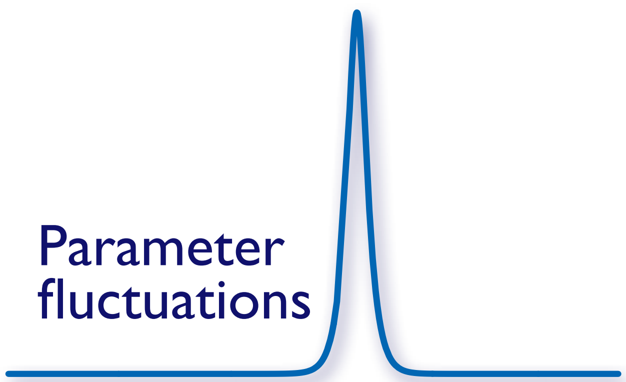
Random forcing

Results: Autocorrelation

- Pulse frequency: $$\langle v_{t+\tau} v_t^* \rangle = \frac{T}{P_0}\left(e^{-\frac{\mu P_0^2}{6}|\tau|} + \pi \int dk k^2 I_k(\tau)\right)$$

Direct term: exponential damping

Parameter fluctuations



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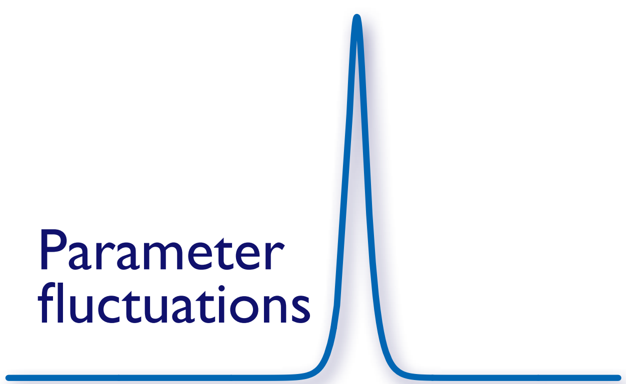
Continuum term: Damped oscillations



$$\boxed{I_k(\tau) = \frac{\mathrm{sech}^2(\pi k/2)}{k^2+1} e^{-\frac{\mu P_0^2(k^2+1)}{8}|\tau|} \cos\left(\frac{P_0^2(k^2+1)}{8}\tau\right)}$$

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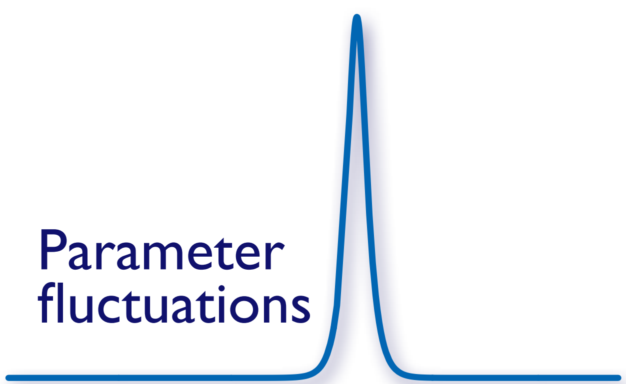
Continuum term: Damped oscillations

Parameter fluctuations



$$I_k(\tau) = \frac{\operatorname{sech}^2(\pi k/2)}{k^2+1} e^{-\frac{\mu P_0^2 (k^2+1)}{8}|\tau|} \cos\left(\frac{P_0^2 (k^2+1)}{8}\tau\right)$$

- Pulse power: $$\langle p_{t+\tau} p_t^* \rangle = \frac{2T}{P_0}\left(\frac{\mathcal{P}}{P_0} e^{-\frac{\mu P_0^3}{4\mathcal{P}}|\tau|} + \frac{\pi}{2}\int dk\, I_k(\tau)\right)$$

Continuum enhancement factor

Results: Autocorrelation

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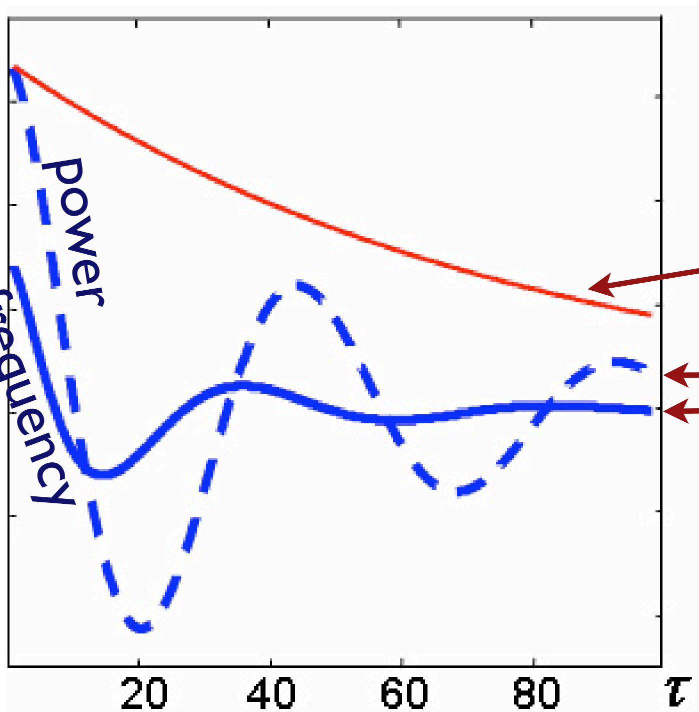

Direct term: exponential damping

Continuum term: Damped oscillations

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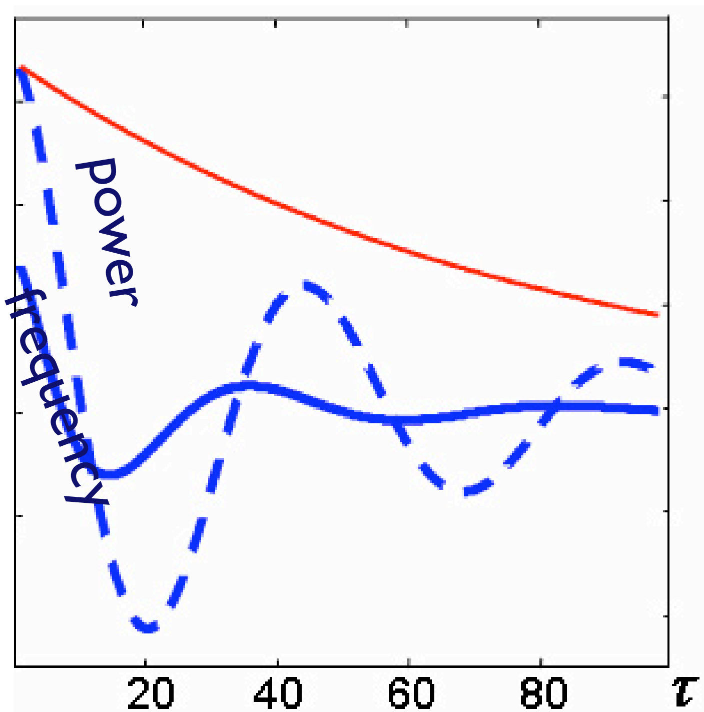
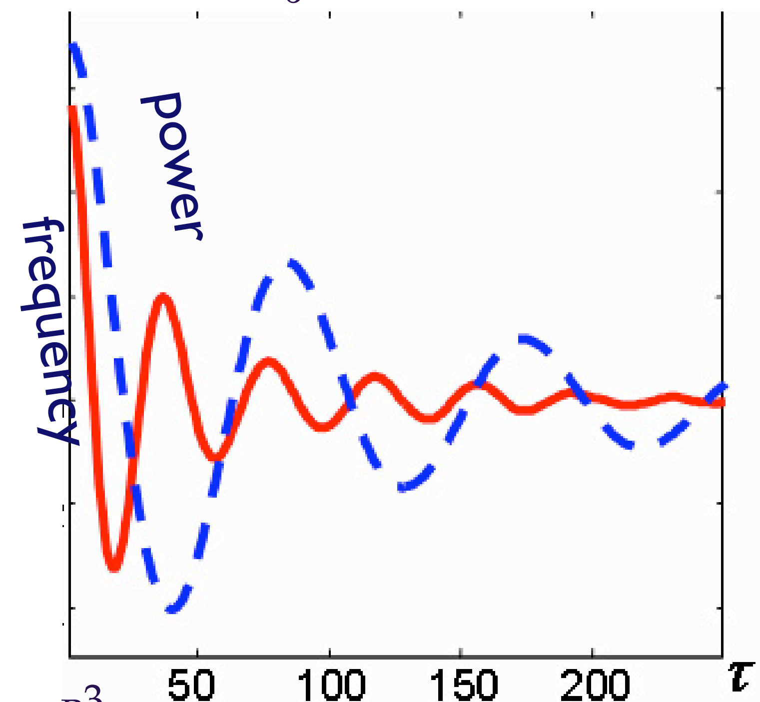


Theory

Numerics

Alternative parameter definition



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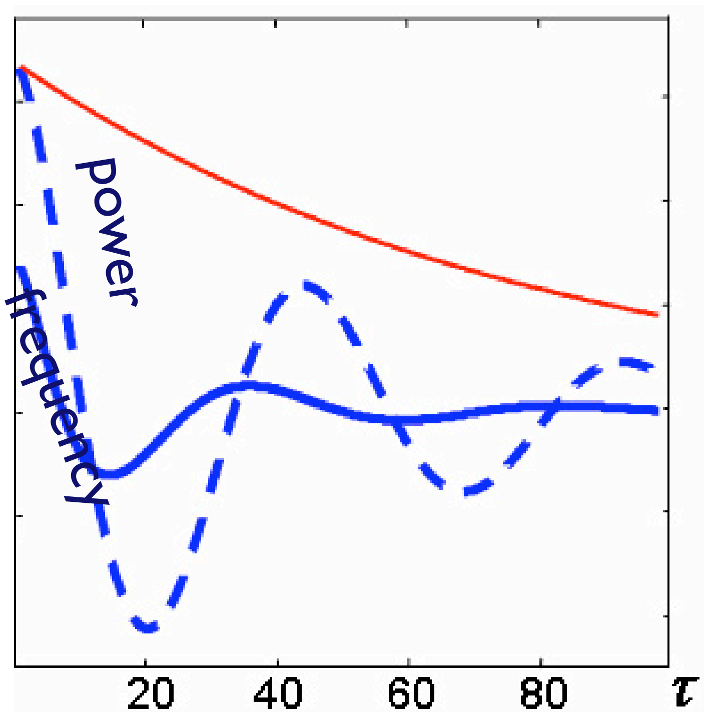

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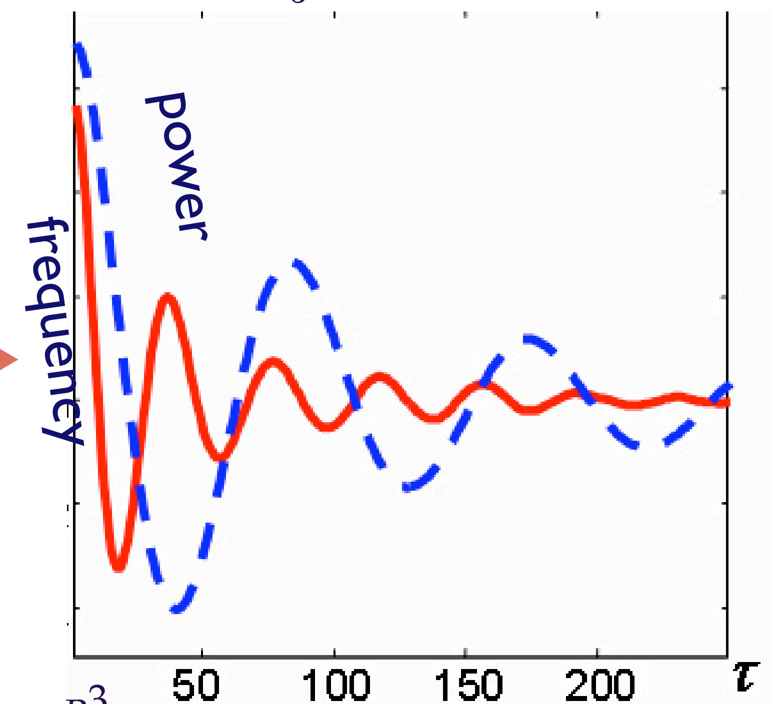

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Nonlinear phase shift

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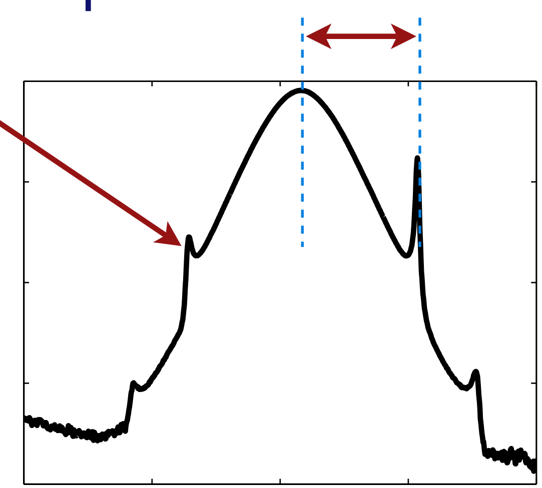
- Compare with *pulse-generated continuum*: Kelly sidebands



Results: Diffusion

Parameter equations

$$\partial_z p = -\mu \frac{P_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p$$

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Frequency- & power-fluctuations driven diffusion

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- Timing jitter $\langle C(z)^2 \rangle = \boxed{12 \frac{T}{P_0^3}} z$ = Haus-Mecozzi jitter
- Phase jitter $\langle \Phi(z)^2 \rangle = \boxed{\frac{T\mathcal{P}^2}{P_0^3}} z$ enhanced by $\left(\frac{\mathcal{P}}{P_0}\right)^2$

Diffusion constants

Summary & conclusions

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- Restoring terms suppressed by continuum: *enhanced* phase jitter

Outlook

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Thank you!