Anderson Localization for the Nonlinear Schrödinger Equation (NLSE): Results and Puzzles

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Experimental Relevance

Nonlinear Optics

Bose Einstein Condensates (BECs)

Competition between randomness and nonlinearity

The Nonlinear Schroedinger (NLS) Equation

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta\left|\psi\right|^2\psi$$

1D lattice version

$$\mathcal{H}_0\psi(x) = -\big(\psi(x+1)+\psi(x-1)\big)+\varepsilon(x)\psi(x)$$

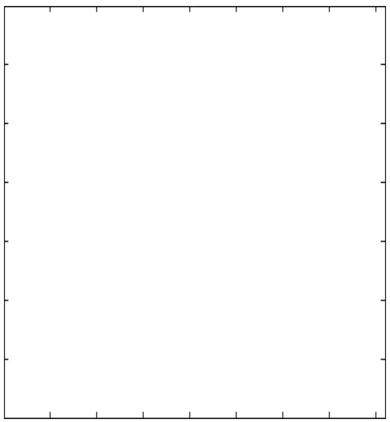
1D continuum version

$$\mathcal{H}_0\psi(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x)+\varepsilon(x)\psi(x)$$

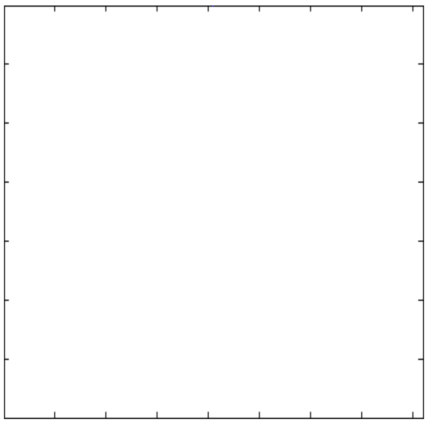
V random ⟶ $\mathcal{H}_0$ Anderson Model

$$i\frac{\partial \psi(x)}{\partial t} = -\left(\psi(x+1)+\psi(x-1)\right)+\varepsilon(x)\psi(x)+\beta\left|\psi(x)^2\right|\psi(x)$$

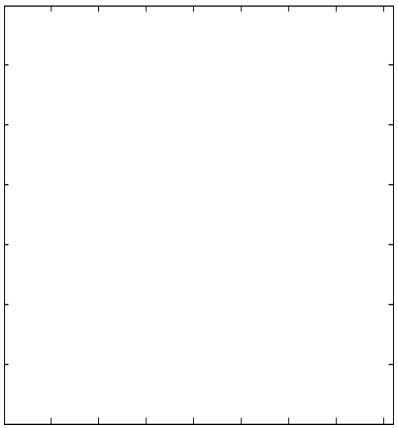
$\beta = 0 \quad \varepsilon_n = 0$



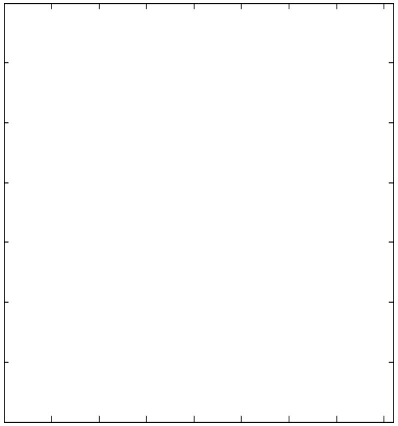
$\beta = 1 \quad \varepsilon_n = 0$



$\beta = -1 \quad \varepsilon_n = 0$



$\beta = 0 \quad \varepsilon_n \neq 0$



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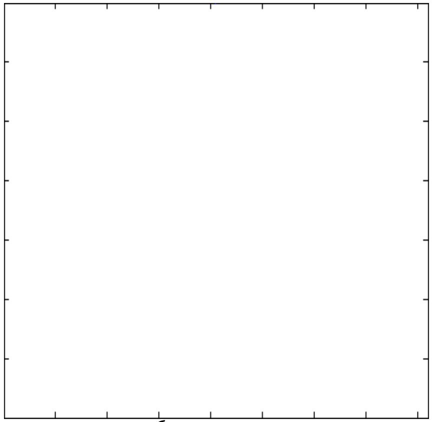

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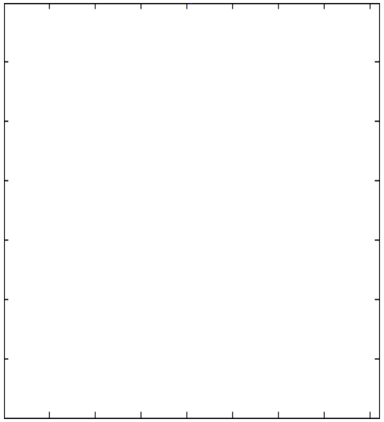
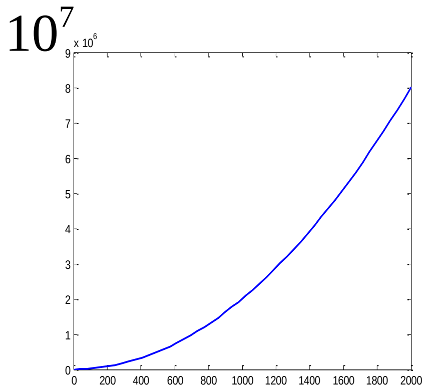


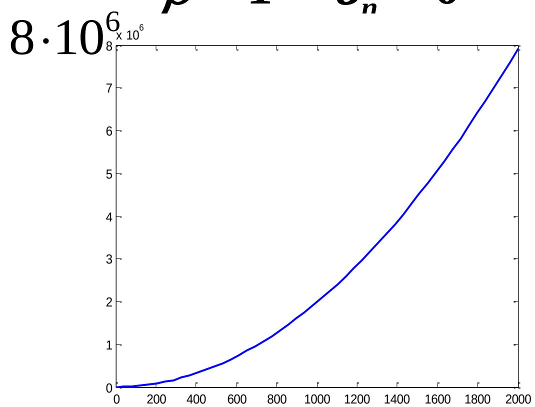
$$\xi \approx 6$$

$$m_2(t) = \sum_n x^2 |\psi(x)|^2$$

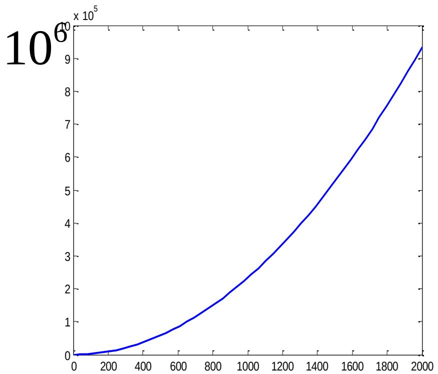
$\beta = 0 \quad \varepsilon_n = 0$



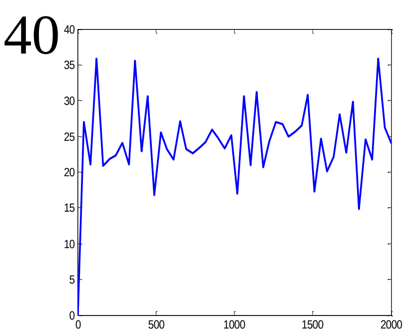
$\beta = 1 \quad \varepsilon_n = 0$



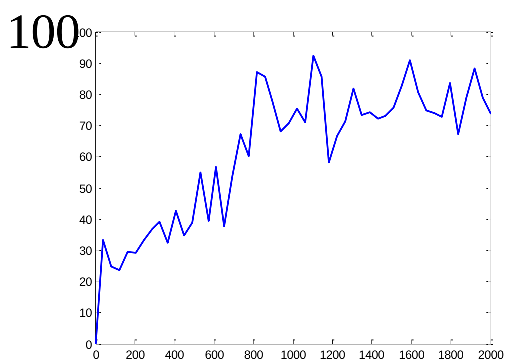
$\beta = -1 \quad \varepsilon_n = 0$



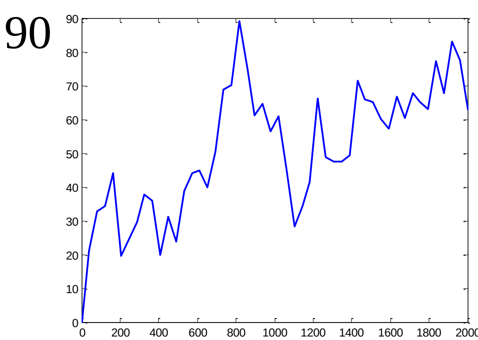
$\beta = 0 \quad \varepsilon_n \neq 0$



$\beta = 1 \quad \varepsilon_n \neq 0$



$\beta = -1 \quad \varepsilon_n \neq 0$



$\beta = 0 \Rightarrow$ localization

Does Localization Survive the Nonlinearity???

Does Localization Survive the Nonlinearity???

- Yes, if there is spreading the magnitude of the nonlinear term decreases and localization takes over.
- No, assume wave-packet width is Δx then the relevant energy spacing is $1/\Delta x$ the perturbation because of the nonlinear term is $\beta|\psi|^2 \approx \beta/\Delta x$ and all depends on β
- No, but does not depend on β
- No, but it depends on realizations

Does Localization Survive the Nonlinearity?

- No, the NLSE is a chaotic dynamical system, will it remain chaotic for all densities??
- No, but localization asymptotically preserved beyond some front that is logarithmic in time

Numerical Simulations

- In regimes relevant for experiments looks that localization takes place
- Spreading for long time (Shepelyansky, Pikovsky, Molina, Kopidakis, Komineas, Flach, Aubry)
- We do not know the relevant space and time scales
- All results in Split-Step
- No control (but may be correct in some range)
- Supported by various heuristic arguments

Pikovsky, Sheplyansky

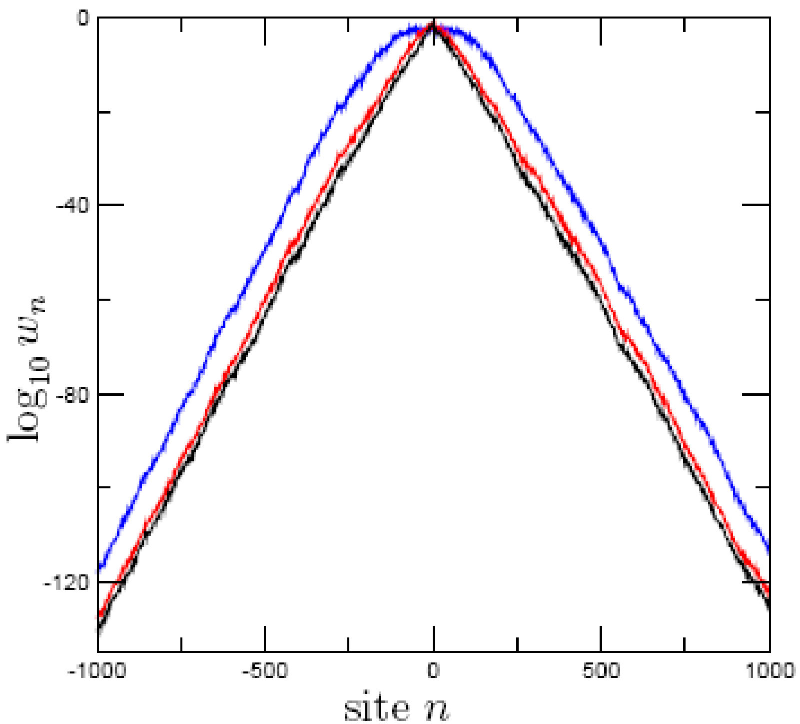


FIG. 2: (color online) Probability distribution w_n over lattice sites n at $W = 4$ for $\beta = 1$, $t = 10^8$ (top blue/solid curve) and $t = 10^5$ (middle red/gray curve); $\beta = 0, t = 10^5$ (bottom black curve; the order of the curves is given at $n = 500$). At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.3$, $\chi \approx 4$. The values of $\log_{10} w_n$ are averaged over the same disorder realizations as in Fig. 1.

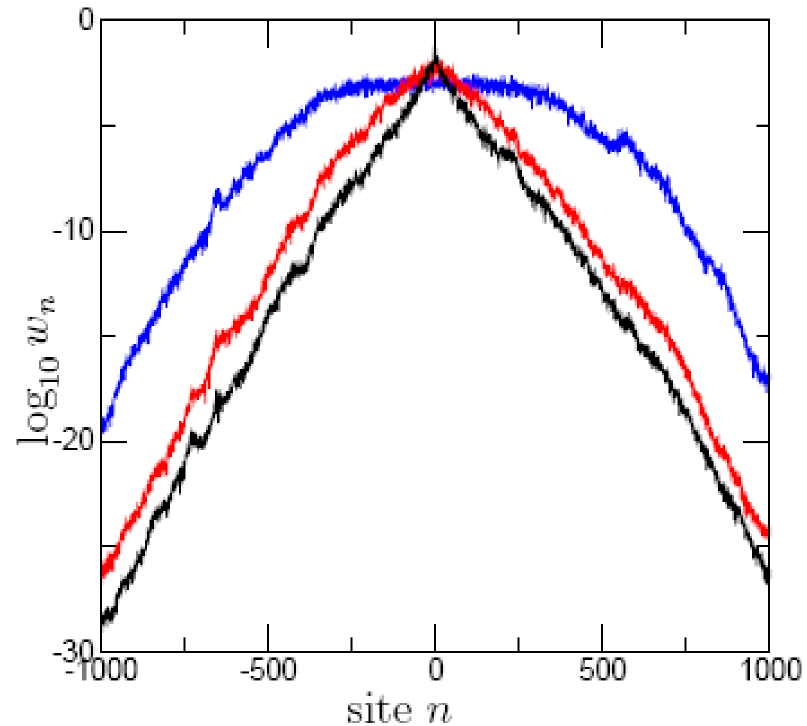


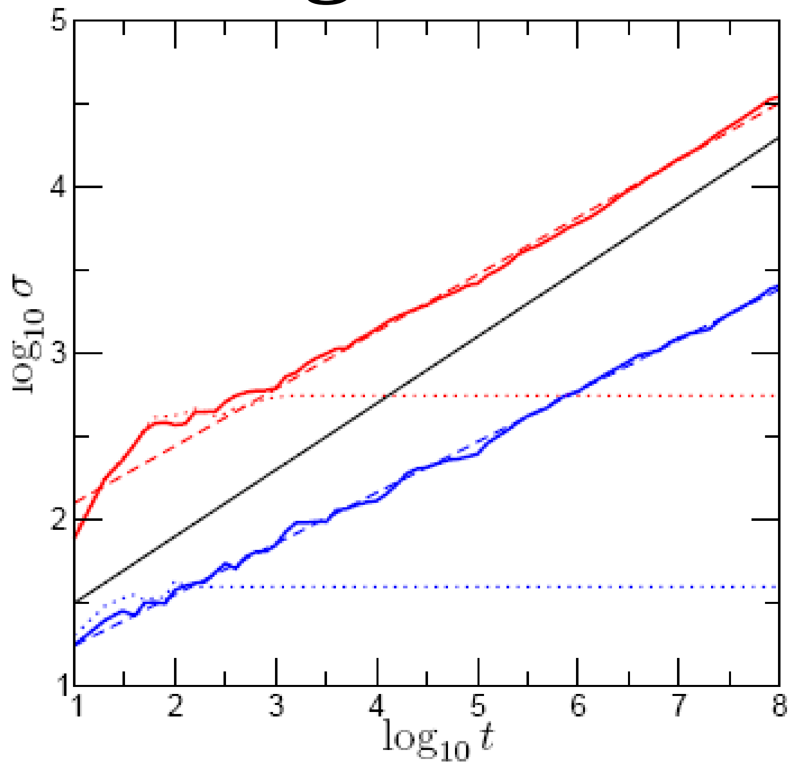
FIG. 3: (color online) Same as in Fig. 2 but with $W = 2$. At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.06$, $\chi \approx -3$. The values of $\ln w_n$ are averaged over the same disorder realizations as in Fig. 1.

spreading sets on. A similar transition occurs for $W = 2$

Slope does not change (contrary to Fermi-Ulam-Pasta)

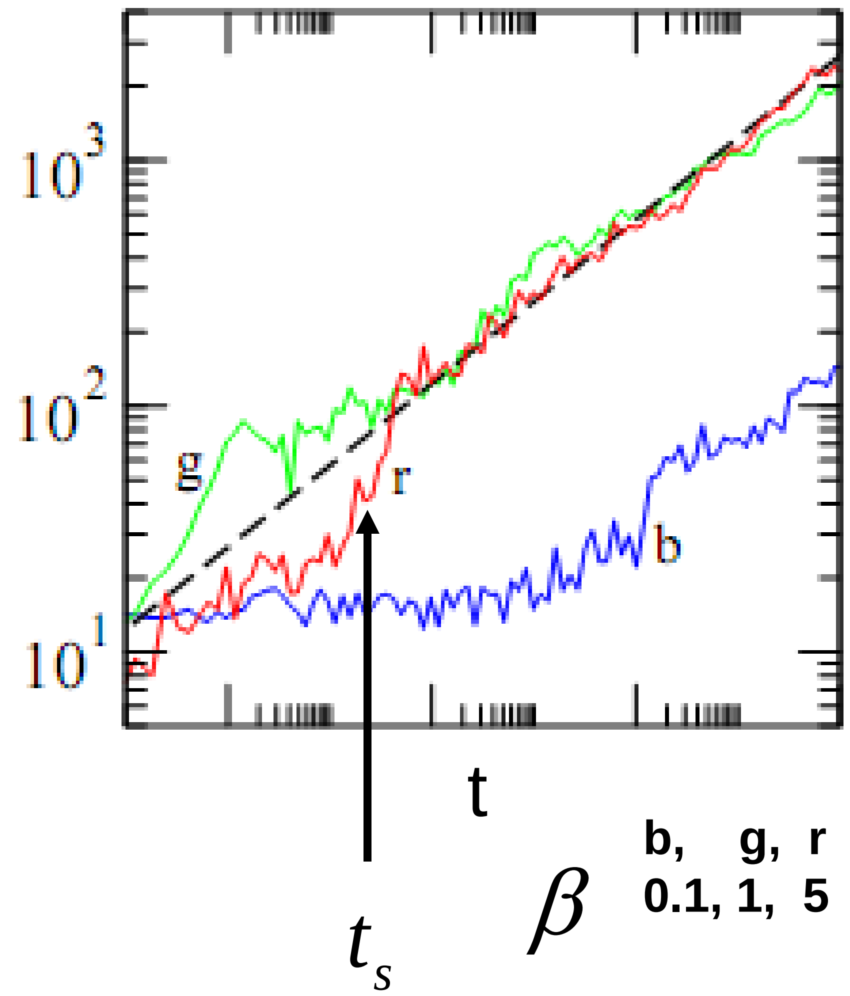
Pikovsky, Shepelyansky

$$\log < x^2 >$$



S.Flach, D.Krimer and S.Skokos

$$\log < x^2 >$$



Effective Noise Theories

- D. Shepeyansky and A. Pikovsky
- Ch. Skokos, D.O. Krimer, S. Komineas and S. Flach

$$\psi(x,t)=\sum_m c_m(t)e^{-iE_m t}u_m(x)$$

$$i\frac{\partial}{\partial t}c_n=\beta\sum_{m_1,m_2,m_3}V_n^{m_1,m_2,m_3}c_{m_1}{}^*c_{m_2}c_{m_3}e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Overlap $$V_n^{m_1,m_2,m_3}=\sum_x u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x)$$

$$\left|V_n^{m_1m_2m_3}\right|\le[const]e^{-\frac{1}{3}\gamma\left(\left|x_n-x_{m_1}\right|+\left|x_n-x_{m_2}\right|+\left|x_n-x_{m_3}\right|\right)}$$

of the range of the localization length ξ

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^{*} c_{m_2} c_{m_3} e^{i(E_n + E_{m_1} - E_{m_2} - E_{m_3})t}$$

Assume $\left|c_{m_1}^2\right| \approx \left|c_{m_2}^2\right| \approx \left|c_{m_3}^2\right| \approx \rho$ **initially** $\left|c_n^2\right| \sim \rho$

$$i\frac{\partial}{\partial t}c_n \approx P\beta\rho^{3/2} f(t)$$

$f(t)$ **Random uncorrelated**

$$\left\langle x^2 \right\rangle \sim t^{1/3}$$

Scaling Properties of Chaos

Arkady Pikovsky

$$i\frac{d}{dt}\psi(x) = -J\left(\psi(x+1)+\psi(x-1)\right)+\varepsilon(x)\psi(x)+\left|\psi(x)\right|^2\psi(x)$$

x integer $1 \le x \le L$

$\psi(x)$ Are dynamical variables

Initial data, nearly homogeneous spreading in space

Growth of deviations

$$\delta\psi(t) \sim \delta\psi(t=0)e^{\lambda t}$$

λ Largest Lyapunov exponent

$\lambda > 0 \longrightarrow$ Chaos

Is it possible that chaos disappears?

Divide chain into intervals of length L_0 Number of intervals $\frac{L}{L_0}$

Assuming independence, if intervals large enough $L_0 \gg \xi$
The probability to be regular:

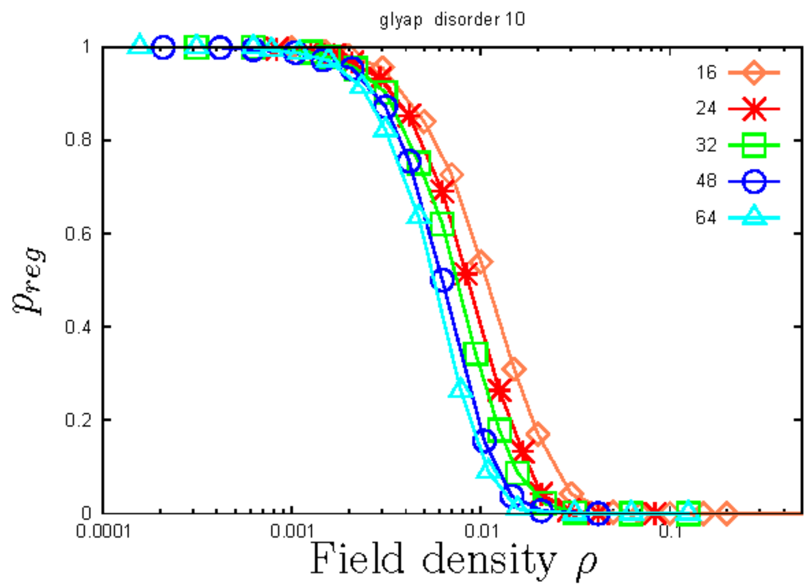
$$p_{reg}(W,\rho,L) = p_{reg}(W,\rho,L_0)^{L/L_0}$$

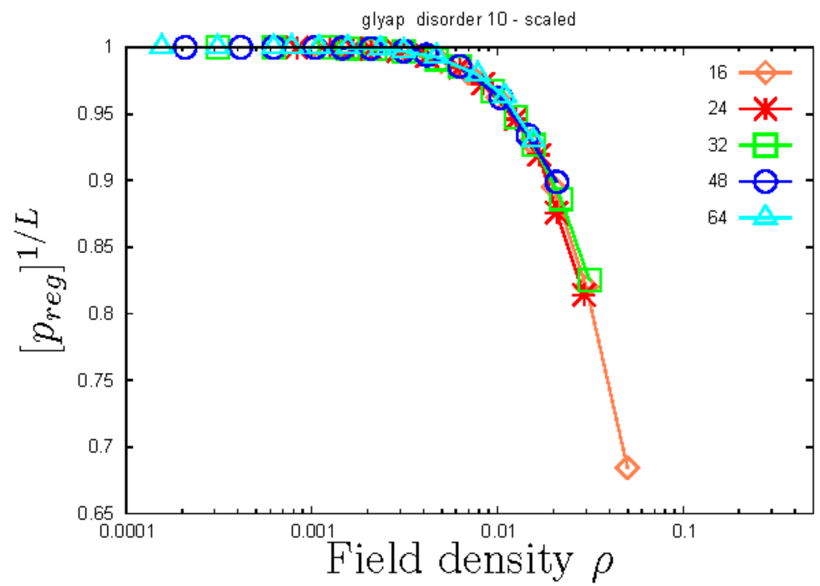
Regularity=all orbits regular

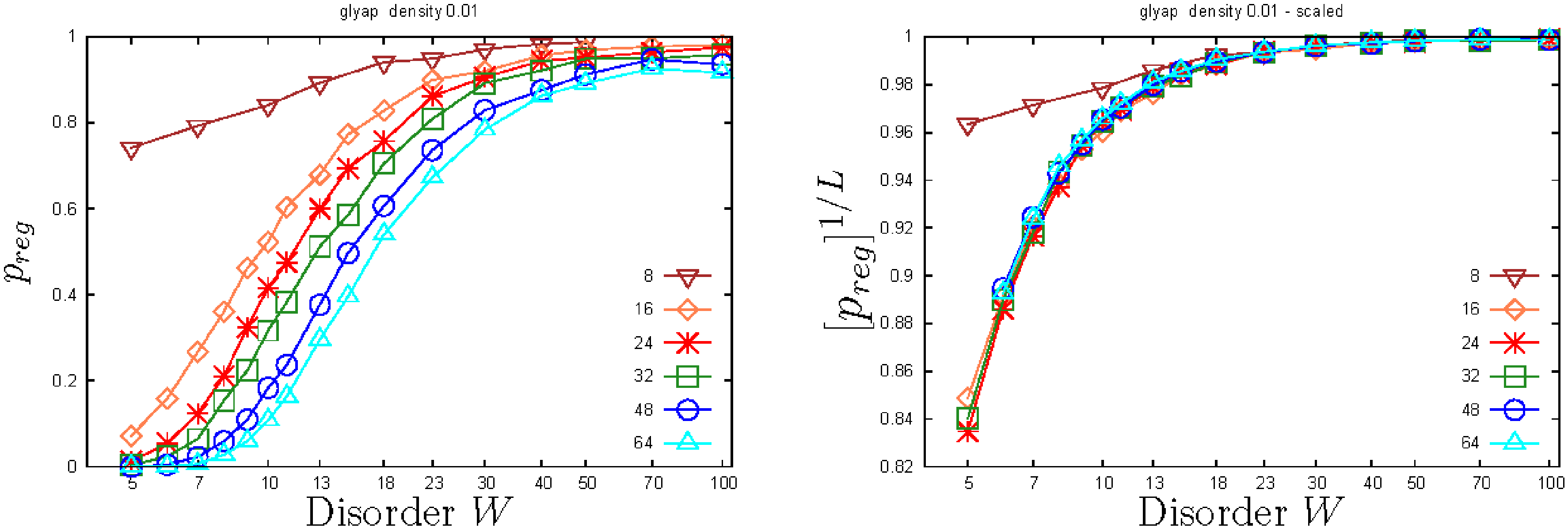
density $\rho = \frac{1}{L}\sum_{x=1}^{L} \left|\psi(x)\right|^2$

$$\bar{p}_{reg}(W,\rho) \equiv p_{reg}(W,\rho,L)^{1/L} = p_{reg}(W,\rho,L_0)^{1/L_0}$$

independent of L


glyap disorder 10
p_{reg}
Field density ρ
16
24
32
48
64


glyap disorder 10 - scaled
$[p_{reg}]^{1/L}$
Field density ρ
16
24
32
48
64


glyap density 0.01
p_{reg}
Disorder W
8
16
24
32
48
64
glyap density 0.01 - scaled
$[p_{reg}]^{1/L}$
Disorder W
8
16
24
32
48
64

Scaling

Define $$Q \equiv \frac{\bar{p}_{reg}}{1-\bar{p}_{reg}} \rightarrow \bar{p}_{reg} = \frac{1}{1+1/Q}$$

Scaling function $$Q = \frac{1}{W^a} q\left(\frac{\rho}{W^b}\right)$$

$$\rho = \frac{1}{L} \sum_{x=1}^{L} |\psi(x)|^2$$

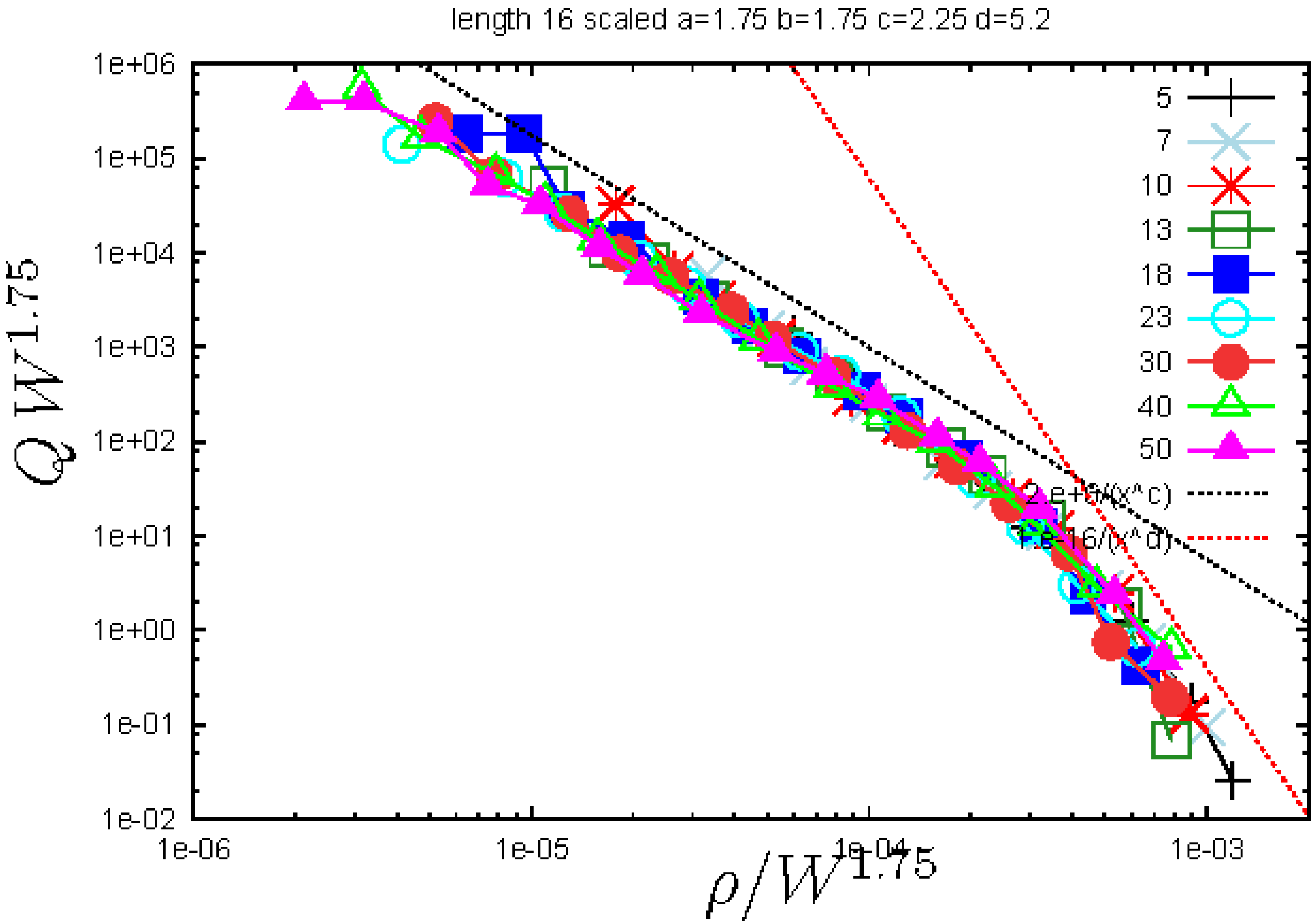
$\sum_{x=1}^{L} |\psi(x)|^2$ conserved

$$\bar{p}_{reg} = \frac{1}{1+1/Q} \approx 1 - \frac{1}{Q} \approx 1 - \rho^{2.25}$$

$$p_{reg} = \bar{p}_{reg}^{L} \rightarrow \ln p_{reg} = L \ln \bar{p}_{reg} \square L\rho^{2.25} \square L^{-1.25}$$

In the limit $L \rightarrow \infty$ $\ln p_{reg} \rightarrow 0 \rightarrow p_{reg} = 1$

Crossover point $\rho \approx 0.005$



length 16 scaled a=1.75 b=1.75 c=2.25 d=5.2
$QW^{1.75}$
$\rho/W^{1.75}$
1e+06
1e+05
1e+04
1e+03
1e+02
1e+01
1e+00
1e-01
1e-02
1e-06
1e-05
1e-04
1e-03
5
7
10
13
18
23
30
40
50
2.e+6/(x^c)
1.e-16/(x^d)

Spreading

No chaos

Localization ?

Perturbation Theory

The nonlinear Schroedinger Equation on a Lattice in 1D

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta\left|\psi\right|^2\psi$$

$$\mathcal{H}_0\psi(x) = -\Big(\psi(x+1)+\psi(x-1)\Big)+\varepsilon(x)\psi(x)$$

$\varepsilon\left(x\right)$ random $\longrightarrow$ $\mathcal{H}_0$ Anderson Model

Eigenstates $\mathcal{H}_0 u_m(x) = E_m u_m(x)$

$$\psi\left(x,t\right) = \sum_m c_m\left(t\right)e^{-iE_m t}u_m\left(x\right)$$

Perturbation theory steps

- Expansion in nonlinearity
- Removal of secular terms
- Control of denominators
- Probabilistic bound on general term
- Control of remainder
- **Use perturbation theory to obtain a numerical solution that is controlled a posteriori**

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^* c_{m_2} c_{m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Overlap $$V_n^{m_1,m_2,m_3} = \sum_x u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x)$$

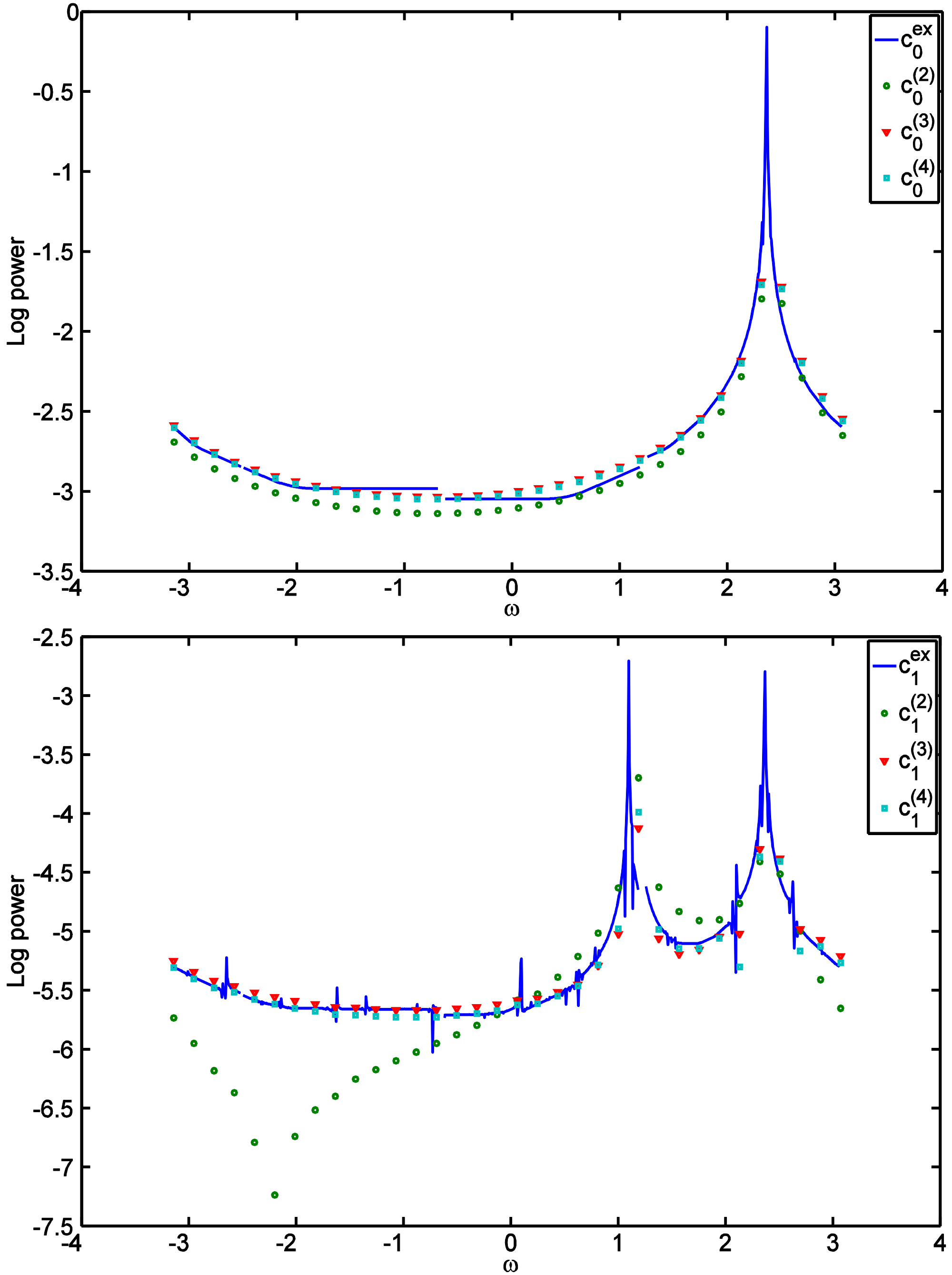
of the range of the localization length ξ

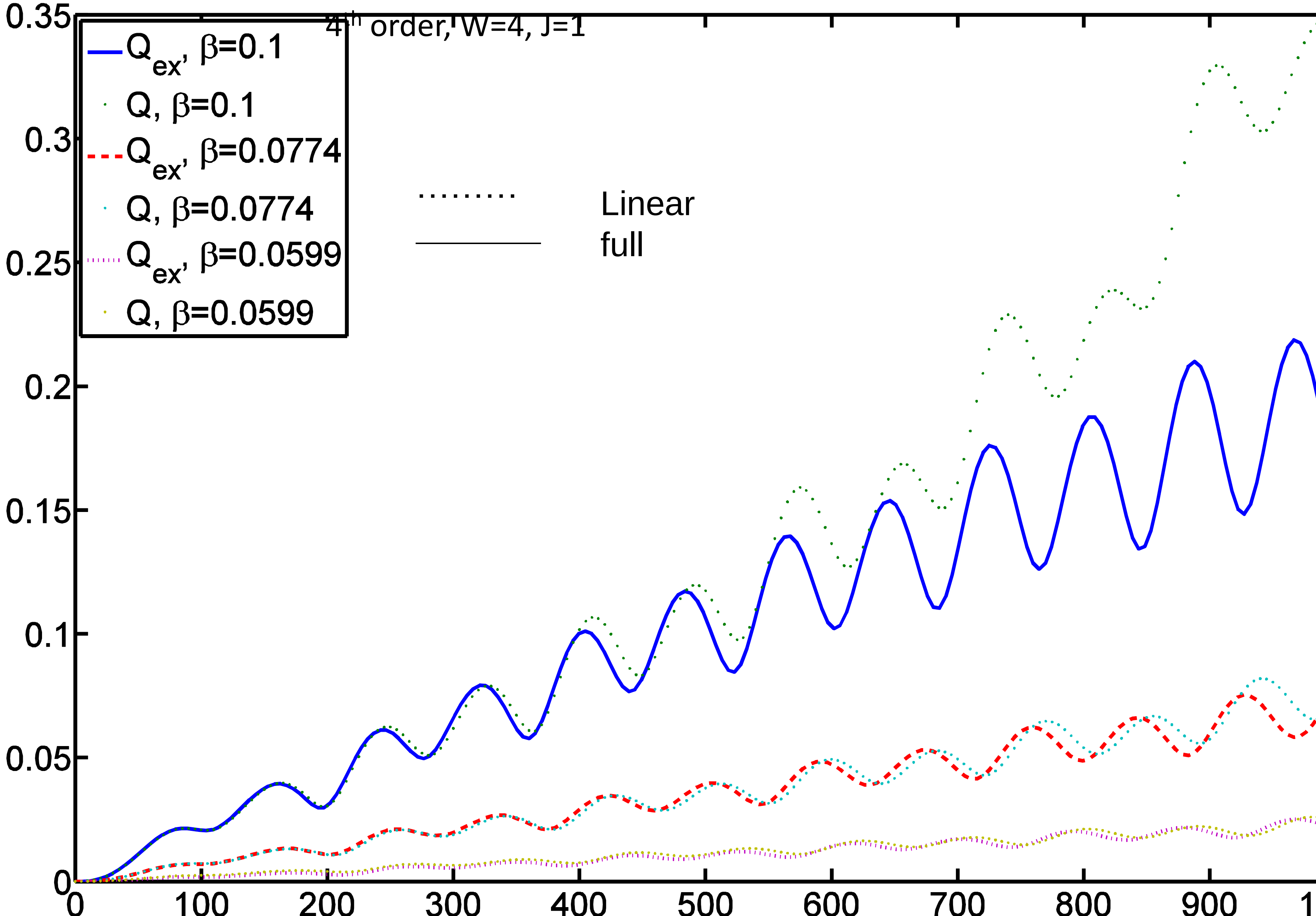
perturbation expansion

$$c_n(t) = c_n{}^{(o)} + \beta c_n{}^{(1)} + \beta^2 c_n{}^{(2)} + ...+ \beta^{N-1} c_n{}^{(N-1)} + \beta^N Q_N(n)$$

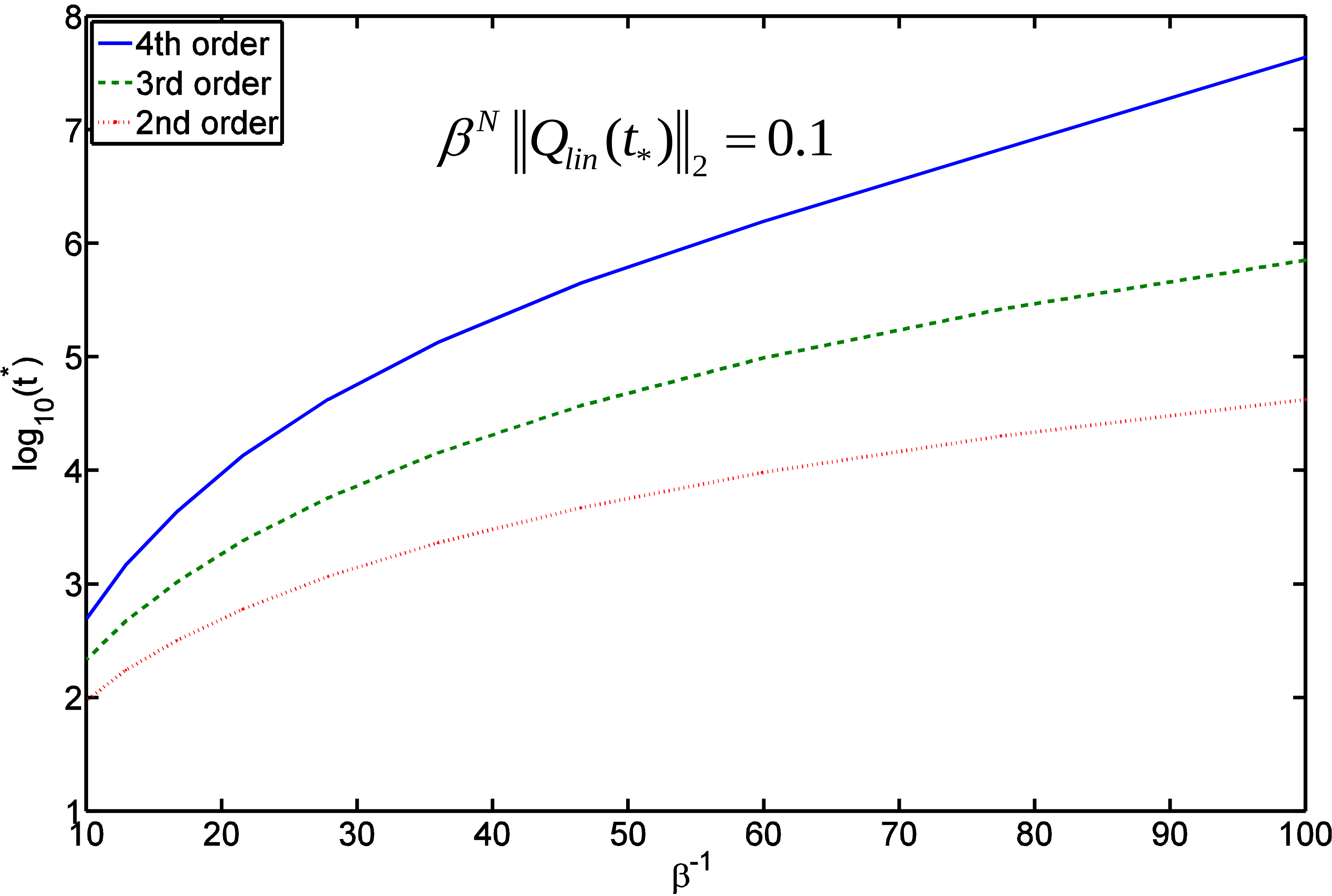
Iterative calculation of $c_n{}^{(l)}$

start at $$c_n^{(0)} = c_n(t=0) = \delta_{n0}$$


Log power
ω
c_0^{ex}
$c_0^{(2)}$
$c_0^{(3)}$
$c_0^{(4)}$
Log power
ω
c_1^{ex}
$c_1^{(2)}$
$c_1^{(3)}$
$c_1^{(4)}$



4th order, W=4, J=1
Q_ex, β=0.1
Q, β=0.1
Q_ex, β=0.0774
Q, β=0.0774
Q_ex, β=0.0599
Q, β=0.0599
Linear
full
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
100
200
300
400
500
600
700
800
900



4th order
3rd order
2nd order
$\beta^N \left\| Q_{lin}(t_*) \right\|_2 = 0.1$
$\log_{10}(t^*)$
β^{-1}
1
2
3
4
5
6
7
8
10
20
30
40
50
60
70
80
90
100

The Bound on the remainder

$$\left|\beta^N Q_N(n)\right| \le A(N,\gamma)\beta^N t e^{-\gamma|x_n|} = A e^{\left(\ln t - N|\ln\beta| - \gamma|x_n|\right)}$$

x_n Localization center of state n

For fixed order and time

$$\lim_{\beta\to 0}\frac{\beta^N Q_N(n)}{\beta^{N-1}} = 0$$ Expansion Asymptotic

One can show that for strong disorder $A(N,\gamma)\xrightarrow[\gamma\to\infty]{}0$

Looks that $A \sim \exp(-\gamma)$ Difficulties in the calculation of A

Front logarithmic in time $\bar{x} \propto \frac{1}{\gamma}\ln t$ For limited time

Localization for $|x| > \bar{x}$

Bound on error

- Solve linear equation for the remainder of order N
- If bounded to time t_0 perturbation theory accurate to that time.
- Order of magnitude estimate $\beta^N t_0 \sim 1$ if asymptotic $\beta^N N! \sim 1$ hence $t_0 \sim N!$ for optimal order (up to constants).
- $t_0 = \beta^{-1/\beta}$ validity time of perturbation theory

Summary Perturbation Theory

1. A perturbation expansion in β was developed
2. Secular terms were removed
3. A bound on the general term was derived
4. Perturbation theory was used to obtain a controlled numerical solution
5. A bound on the remainder was obtained, indicating that the series is asymptotic.
6. For limited time tending to infinity for small nonlinearity, front logarithmic in time $\bar{x} \propto \ln t$
7. Improved for strong disorder

Emerging Picture

- For small nonlinearity initially no spreading
- For strong nonlinearity some part does not spread
- For some nonlinearity wide regime of sub-diffusion
- Asymptotic spreading at most logarithmic:

a. perturbation theory
b. rigorous results in the limit of strong disorder

- Unlikely that sub-diffusion continues forever:

a. scaling theory
b. Effective noise “theories”

Coherent picture for various regimes?