Nonuniform SINR+Voroni Diagrams are Effectively Uniform

Erez Kantor * Zvi Lotker †‡ Merav Parter §‡ David Peleg §‡

August 12, 2015

Abstract

This paper concerns the behavior of an *SINR diagram* of wireless systems, composed of a set S of n stations embedded in $\mathbb{R}^d$, when restricted to the corresponding *Voronoi diagram* imposed on S. The diagram obtained by restricting the SINR zones to their corresponding Voronoi cells is referred to hereafter as an *SINR+Voronoi diagram*.

The study of SINR+Voronoi diagrams is motivated by the following two facts. (1) Uniform SINR diagrams (where all stations transmit with the same power) are simple and nicely structured. In particular, the reception zone of each station is convex and "fat"; this can be used to devise an efficient algorithm for the fundamental problem of point location [3]. (2) In contrast, nonuniform SINR diagrams (where transmission energies are arbitrary) might be complex; the reception zone of each station might be fractured and its boundary might contain many singular points [9]. This makes it harder to understand the geometry of nonuniform SINR diagrams, as well as to design efficient point location algorithms for this setting.

In this paper, we establish the (perhaps surprising) fact that a nonuniform SINR+Voronoi diagram is topologically almost as nice as a uniform SINR diagram. In particular, it is convex and effectively[1] fat. This holds for every power assignment, every path-loss parameter α and every dimension $d \geq 1$. The convexity property also holds for every SINR threshold $\beta > 0$, and the affective fatness holds for any $\beta > 1$. These fundamental properties provide a theoretical justification to engineering practices basing zoneal tessellations on the Voronoi diagram, and helps to explain the soundness and efficacy of such practices.

We then consider two algorithmic applications. The first concerns the *Power Control with Voronoi Diagram* (PCVD) problem, where given n stations embedded in some polygon $\mathcal{P}$, it is required to find the power assignment that optimizes the SINR threshold of the transmission station s_i for any given reception point $p \in \mathcal{P}$ in its Voronoi cell $\text{VOR}(s_i)$. The second application is approximate point location; we show that for SINR+Voronoi zones, this task can be solved considerably more efficiently than in the general non-uniform case.

*MIT, CSAIL. E-mail: erezk@csail.mit.edu.

†Department of Communication Systems Engineering, Ben Gurion University, Beer-Sheva, Israel. E-mail: zvilo@cse.bgu.ac.il.

‡Supported in part by the Israel Science Foundation (grant 894/09).

§Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel. E-mail: {merav.parter,david.peleg}@ weizmann.ac.il.

[1]in the sense that its fatness measure does not depend on the number of stations n but only on parameters typically bounded by a constant.

1 Introduction

1.1 Background and motivation

A common method for designing a cellular or wireless network in a plane is by computing the Voronoi diagram of the base-stations, and making each base-station responsible for its own Voronoi cell. This choice is natural, since it ensures that the distance from every point p in the plane to the station responsible for it is minimal. Yet what affects the performance of a wireless network is not just the distance. Rather, reception at a given point in a given time is governed by a complex relationship between the reception point and the set of stations that transmit at that time. This relationship is described schematically by the SINR formula, which also dictates the reception zones around each transmitted station. Hence the areas in the intersection between SINR reception regions and their corresponding Voronoi cells deserve particular attention, and are the focus of the current paper.

We consider the *Signal to Interference-plus-Noise Ratio (SINR)* model, where given a set of stations $S = \{s_0, \ldots, s_{n-1}\}$ in $\mathbb{R}^d$ concurrently transmitting with power assignment ψ, and background noise N, a receiver at point $p \in \mathbb{R}^d$ successfully receives a message from station s_i if and only if $\mathrm{SINR}(s_i, p) \geq \beta$, where $\mathrm{SINR}(s_i, p) = \frac{\psi_i \cdot \mathrm{dist}(s_i,p)^{-\alpha}}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}(s_j,p)^{-\alpha} + N}$ for constants α and $\beta \geq 1$, and where dist() denotes Euclidean distance.

To model the reception zones we use the convenient representation of an *SINR diagram*, introduced in [3], which partitions the plane into n reception zones, one per station, and a complementary zone where no station can be heard. The topology and geometry of SINR diagrams was studied in [3] in the relatively simple setting of *uniform power*, where all stations transmit with the same power level. It was shown therein that uniform SINR diagrams are particularly simple: the reception zone of each station is convex, fat and strictly contained inside the corresponding Voronoi cell.

SINR diagrams under the general *nonuniform* setting (i.e., with arbitrary power assignments) were studied in [9]. The topological features of general SINR diagrams turn out to be much more complicated than in the uniform case, even for small networks. In particular, the reception zones are not necessarily fat, convex or even connected, and their boundaries might contain many singular points.

In this paper, we explore the behavior of the reception zones of SINR diagrams when restricted to Voronoi diagrams. The resulting diagram, referred to as *SINR+Voronoi* diagram, consists of n reception zones, one per station, obtained by the intersection of the SINR reception zones with their corresponding Voronoi cells. Studying SINR+Voronoi diagrams is motivated by the complexity of general nonuniform SINR zones and, perhaps more importantly, by the abundant usage of hexagonal networks in practice; cellular networks are commonly designed as hexagonal networks, where each node serves as a base-station to which mobile users must connect. A mobile user is normally connected to the nearest base-station, hence each base-station serves all users that are located inside its hexagonal grid cell (which is in fact its Voronoi cell). Due to the disk shape of the sensing range of the sensor devices, using a hexagonal tessellation topology is the most efficient way to cover the whole sensing area, and indeed many routing, location management and channel assignment protocols are based on it [6, 12, 13, 14, 15].
It is thus intriguing to ask whether the reception zones of *nonuniform* SINR diagrams enjoy some desirable properties (e.g., assume a convenient form) when *restricted* to their corresponding Voronoi cells.

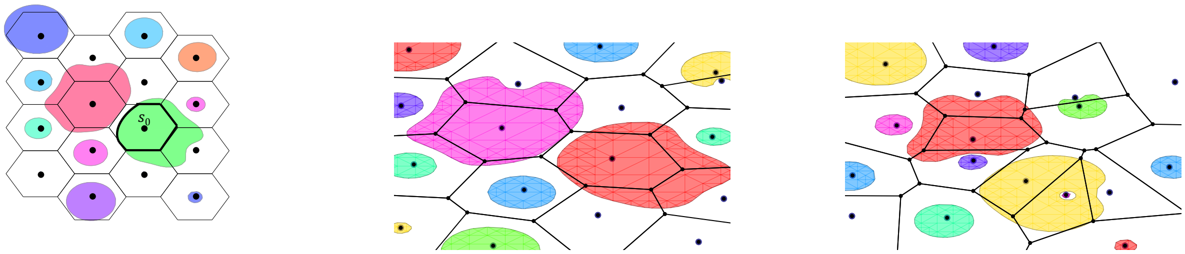

Figure 1: The overlay of an SINR diagram of a nonuniform wireless network on the corresponding Voronoi diagram. (a) Hexagonal Voronoi cells; the intersection between the reception region of station s_0 and the Voronoi cell around it is highlighted in bold. (b) Slight random perturbation to a hexagonal network. (c) Random positions.

In this paper, it is shown that while in general the reception zones in the nonuniform setting might be fractured and their boundaries might contain many singular points, the restriction of a reception zone to its Voronoi cell (e.g., hexagonal cell in the grid) behaves almost as nice as *uniform* zones: it is *convex*, and its fatness measure depends only on parameters typically bounded by a constant, and in particular is independent of the number of stations in the network.

For an illustration see the reception zone of station s_0 in Figure 1(a). These fundamental properties provide a theoretical justification to engineering practices basing regional tessellations on the Voronoi diagram, and helps to explain the soundness and efficacy of such practices.

To prove convexity, we extend the proof for the uniform setting of [3] to the nonuniform setting[2]. Apart from the theoretical interest, this result is of considerable practical significance, as obviously, having a convex reception zone inside each hexagonal cell may ease the development of protocols for various design and communication tasks such as scheduling, topology control and connectivity. We note that convexity within a Voronoi cell is important also in the mobile setting where no fixed tessellation can be assumed. For example, in the setting of Vehicular ad-hoc network (VANET) [17], the stations are mobile but each user is still mapped to the closest base-station. Hence, although the hexagonal tessellation is no longer preserved, the convexity within the (dynamic) Voronoi tessellation is still relevant (for an illustration, see Fig. 1(b)-(c)).

As an application for the convexity property, we consider the problem where one wishes to cover the entire area of a given bounded polygon $\mathcal{P}$ by using a base-station network embedded in $\mathcal{P}$. One natural way to do that is by assigning each base-station an area of coverage. Usually the base-station needs to cover the area of its Voronoi cell up to where it intersects with $\mathcal{P}$. Assuming the power with which each base-station transmits can be controlled, it is desirable to increase the SINR ratio as much as possible in order to increase the capacity of the cellular network. The problem of determining the transmission energy of each base-station so as to maximize the capacity of the entire network is called the *Power Control Voronoi Diagram* (PCVD) problem. We show that although PCVD is a non-convex and non-discrete problem, it can be solved in a nearly optimal manner.

Our algorithm is especially useful in the mobile setting where the positions of base-stations vary

[2]Note that in the uniform setting too, convexity is guaranteed only inside the Voronoi cell, but since the entire reception zone is restricted to the Voronoi cell, this implies that the entire zone is convex. In contrast, in the nonuniform setting, the reception zone of a station with a high transmission energy might exceed its Voronoi cell.

with time. This scenario can happen in sudden-onset disasters and ad-hoc vehicle networks, since in these cases, the network structure is not fixed and it is not clear how to divide the coverage areas between the base-stations. Although it is natural to use the Voronoi diagram, it is not clear how to assign the transmission energies in a way that guarantees a full coverage of the area of interest. The solution proposed in this paper for this problem has the advantage that it can adapted to a dynamic setting quite efficiently since it depends upon the Voronoi tessellation that can be maintained efficiently in a dynamic setting [8, 5]. By exploiting the convexity property in Voronoi cells, we propose a discrete equivalent formulation of the PCVD problem. Specifically, thanks to the convexity guarantee, we show that it is sufficient to insist on achieving the optimal threshold β only on the vertex set of each Voronoi cell (where unbounded Voronoi cells are bounded by using a bounding polygon $\mathcal{P}$ that contains the entire coverage area). Computing power assignment for maximizing the coverage within Voronoi cells has been considered also in [16] from the game theoretic point of view; yet no analytic result has been known so far for this problem.

We then turn to consider the fatness property. In [9], it was shown that the fatness of nonuniform zone can be bounded by some function of the maximum transmission power $\psi_{\max}$, the ambient noise N, the SINR threshold β, the path-loss exponent α, the distance κ to the closest interfering station and the *number* of stations in the network. The SINR+Voronoi zones are shown to have a fatness bound that is *independent* of n. In particular, since the network parameters α, β, κ, N and $\psi_{\max}$ are bounded in practice (i.e., unlike the number of stations), the SINR+Voronoi zones are effectively fat. Finally, using [4], the convexity and the improved fatness bound imply an efficient approximate point location scheme for SINR+Voronoi zones whose preprocessing time and memory requirements are significantly more efficient than those obtained in [9]. For a recent work on batched point location tasks, see recent work of [1].

1.2 Geometric notions and wireless networks

Geometric notions. We consider the d-dimensional Euclidean space $\mathbb{R}^d$ (for $d \in \mathbb{Z}_{\geq 1}$). The *distance* between points p and point q is denoted by $\text{dist}(p,q) = \|q - p\|$. Denote the *ball* of radius r centered at point $p \in \mathbb{R}^d$ by $B^d(p,r) = \{q \in \mathbb{R}^d \mid \text{dist}(p,q) \leq r\}$. Unless stated otherwise, we assume the 2-dimensional Euclidean plane, and omit d. The basic notions of open, closed, bounded, compact and connected sets of points are defined in the standard manner.

We use the term *zone* to describe a point set with some "niceness" properties. Unless stated otherwise, a zone refers to the union of an open connected set and some subset of its boundary. It may also refer to a single point or to the finite union of zones.

The point set P is said to be *star-shaped* with respect to point $p \in P$ if the line segment $\overline{p\,q}$ is contained in P for every point $q \in P$. In addition, P is said to be *convex* if it is star-shaped with respect to any point $p \in P$, see [7].

For a bounded zone $Z \neq \emptyset$ and an internal $p \in Z$, denote the maximal and minimal diameters of Z w.r.t. p by $\delta(p,Z) = \sup\{r > 0 \mid Z \supseteq B(p,r)\}$ and $\Delta(p,Z) = \inf\{r > 0 \mid Z \subseteq B(p,r)\}$, and define the *fatness parameter* of Z *with respect to* p to be $\varphi(p,Z) = \Delta(p,Z)/\delta(p,Z)$. The zone Z is said to be *fat* with respect to p if $\varphi(p,Z)$ is bounded by some constant.

Wireless networks and SINR Diagrams. We consider a wireless network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$, where $d \in \mathbb{Z}_{\geq 1}$ is the dimension, $S = \{s_0, s_1, \ldots, s_{n-1}\}$ is a set of transmitting $n \geq 2$ *radio stations* embedded in the d-dimensional space, ψ is an assignment of a positive real *transmitting power* ψ_i to each station s_i, $N \geq 0$ is the *background noise*, $\beta \geq 0$ is a constant *reception threshold*, and

$\alpha > 0$ is the *path-loss parameter*. The *signal to interference & noise ratio (SINR)* of s_i at point p is defined as

$$\mathrm{SINR}_{\mathcal{A}}(s_i, p) \ = \ \frac{\psi_i \cdot \mathrm{dist}(s_i, p)^{-\alpha}}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}(s_j, p)^{-\alpha} + N} \ . \tag{1}$$

Observe that $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$ is always positive since the transmission powers and the distances of the stations from p are always positive and the background noise is non-negative. In certain contexts, it may be more convenient to consider the reciprocal of the SINR function,

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_i, p) \ = \ \frac{1}{\psi_i}\big(\sum_{j \neq i} \psi_j \left(\frac{\mathrm{dist}(s_i, p)}{\mathrm{dist}(s_j, p)}\right)^{\alpha} + N \cdot \mathrm{dist}(s_i, p)^{\alpha}\big).$$

When the network $\mathcal{A}$ is clear from the context, we may omit it and write simply $\mathrm{SINR}(s_i, p)$. The fundamental rule of the SINR model is that the transmission of station s_i is received correctly at point $p \notin S$ if and only if its signal to noise ratio at p is not smaller than the reception threshold of the network, i.e., $\mathrm{SINR}(s_i, p) \geq \beta$. In this case, we say that s_i is *heard* at p. We refer to the set of points that hear station s_i as the *reception zone* of s_i, defined as

$$\mathcal{H}_{\mathcal{A}}(s_i) \ = \ \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}_{\mathcal{A}}(s_i, p) \geq \beta\} \cup \{s_i\} \ .$$

(Note that $\mathrm{SINR}(s_i, \cdot)$ is undefined at points in S and in particular at s_i itself, and that $\mathcal{H}_{\mathcal{A}}(s_i)$ is not is not necessarily connected or restricted to the Voroni cell $\mathrm{VOR}(s_i)$). The *null zone* is the set of points that hear no station $s_i \in S$ (due to the background noise and interference), $\mathcal{H}_{\mathcal{A}}(\emptyset) = \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}(s_i, p) < \beta, \ \forall s_i \in S\}$. An SINR diagram $\mathcal{H}(\mathcal{A}) = \{\mathcal{H}_{\mathcal{A}}(s_i), \ \ 0 \leq i \leq n-1\} \cup \{\mathcal{H}_{\mathcal{A}}(\emptyset)\}$ is a "reception map" partitioning the plane into the stations reception zones and the null zone. The following important technical lemma from [3] will be useful in our later arguments.

Lemma 1.1 *[3] Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a mapping consisting of rotation, translation, and scaling by a factor of $\sigma > 0$. Consider some network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$ and let $f(\mathcal{A}) = \langle d, f(S), \psi, N/\sigma^2, \beta, \alpha \rangle$, where $f(S) = \{f(s_i) \mid s_i \in S\}$. Then f preserves the signal to noise ratio, namely, for every station s_i and for all points $p \notin S$, we have* $\mathrm{SINR}_{\mathcal{A}}(s_i, p) \ = \ \mathrm{SINR}_{f(\mathcal{A})}(f(s_i), f(p)).$

Avin et al. [3] discuss the relationships between an SINR diagram on a set of stations S with *uniform* transmission powers and the corresponding *Voronoi diagram* on S. Specifically, it is shown that the n reception zones $\mathcal{H}_{\mathcal{A}}(s_i)$ around each point s_i are strictly contained in the corresponding Voronoi cells $\mathrm{VOR}(s_i)$ where

$$\mathrm{VOR}(s_i) = \{p \in \mathbb{R}^d \mid \mathrm{dist}(s_i, p) \leq \mathrm{dist}(s_j, p) \text{ for any } j \neq i\} \ . \tag{2}$$

In contrast, the reception zone of a nonuniform SINR diagram is *not* necessarily contained within the Voronoi cell of the corresponding station (e.g., a strong station with high transmission energy may be successfully received in zones outside its Voronoi cell). Kantor et al. [9] showed that nonuniform SINR diagrams are related to a *weighted* variant of Voronoi diagrams [2].

SINR+Voronoi Diagrams. Consider a wireless network $\mathcal{A} = \langle d, S, \bar{\psi}, N, \beta, \alpha \rangle$. Let $\mathrm{VOR}(s_i)$ be the Voronoi cell of station s_i (see Eq. (2)). Define $\mathcal{VH}_{\mathcal{A}}(s_i)$ be the reception zone of s_i restricted to its Voronoi cell, where

$$\mathcal{VH}_{\mathcal{A}}(s_i) = \mathcal{H}_{\mathcal{A}}(s_i) \cap \mathrm{VOR}(s_i) \ .$$

The SINR+Voronoi diagram consists of the n restricted reception zones $\mathcal{VH} = \langle \mathcal{VH}_{\mathcal{A}}(s_0), \ldots, \mathcal{VH}_{\mathcal{A}}(s_{n-1}) \rangle$.

2 Convexity of SINR+Voronoi Zones

Without loss of generality, throughout we fix a station s_0 and show the following (for an illustration see Fig. 2).

Theorem 2.1 *For every wireless network* $\mathcal{A} = \langle d, S, \psi, N \geq 0, \beta > 0, \alpha \rangle$, $\mathcal{VH}_{\mathcal{A}}(s_0)$ *is convex.*

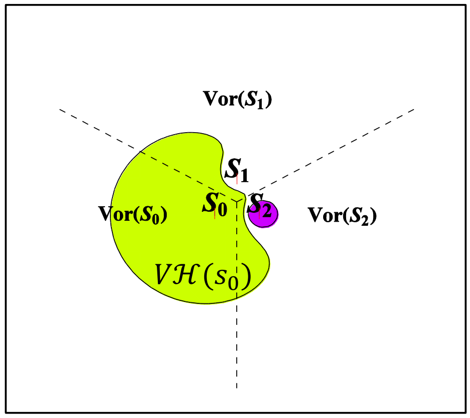

Figure 2: The reception region of s_2 is non-convex but it is convex restricted to its Voronoi cell.

The following technical lemma from [11] plays a key role in our analysis. Denote the origin point by $q = (0, 0)$, let $p_L = (1, 0)$, $p_R = (-1, 0)$ and define $\rho_i = \text{dist}^2(s_i, q)$, for every $i = 0, ..., n-1$.

Lemma 2.2 ([11]) *Let* $\mathcal{A}$ *be a noise-free network* ($N = 0$) *and let* $q \notin S$. *Then*
$\max\{\text{SINR}_{\mathcal{A}}^{-1}(s_0, p_L) \ , \ \text{SINR}_{\mathcal{A}}^{-1}(s_0, p_R)\} \ \geq \ \sum_{i=1}^{n-1} \frac{\psi_i}{\psi_0} \cdot \left(\frac{\rho_0+1}{\rho_i+1}\right)^{\alpha/2}$.

Our proof scheme for Lemma 2.1 is as follows. For simplicity, consider the two-dimensional case. Using [3], the proof naturally extends to any dimension $d \geq 2$. Consider a pair of reception points $p_1, p_2 \in \mathcal{VH}_{\mathcal{A}}(s_0)$. We classify such pairs into two types. The first type is where $s_0 \in \overline{p_1\, p_2}$. This case is handled in Lemma 2.3, where it is shown that $\mathcal{VH}_{\mathcal{A}}(s_0)$ is *star-shaped* with respect to s_0. The complementary case, where $s_0 \notin \overline{p_1\, p_2}$ is handled in two steps. First, in Lemma 2.4, we consider the simplified case where there is no background noise (i.e., $N = 0$) and use Lemma 2.2 to establish the claim. Finally, we consider the general noisy case where $N > 0$ and establish Theorem 2.1.

Lemma 2.3 $\mathcal{VH}_{\mathcal{A}}(s_0)$ *is star-shaped with respect to* s_0.

Proof: In fact, we prove a slightly stronger assertion. Consider some point $p \in \text{Vor}(s_0)$. We show that $\text{SINR}(s_0, q) > \text{SINR}(s_0, p)$ for all internal points q in the segment $\overline{s_0\, p}$. By Lemma 1.1, we may assume without loss of generality that $s_0 = (0, 0)$ and $p = (-1, 0)$. Consider some station s_i, $i > 0$. If s_i is not located on the positive half of the horizontal axis, then it can be relocated to a new location s_i' on the positive half of the horizontal axis by rotating it around p so that

$\text{dist}(s'_i, p) = \text{dist}(s_i, p)$ and $\text{dist}(s'_i, q) \leq \text{dist}(s_i, q)$ for all points $q \in \overline{s_0\, p}$ (see Fig. 3). This process can be repeated with every station s_i, $i > 0$, until all interfering stations $s_i \neq s_0$ are located on the positive half of the horizontal axis without decreasing the interference at any point $q \in \overline{s_0\, p}$. Therefore it is sufficient to establish the assertion under the assumption that $s_i = (a_i, 0)$, where $a_i > 0$, for every $i > 0$. Let $q = (-x, 0)$ for some $x \in (0, 1]$. To show that $\text{SINR}(s_0, q) > \text{SINR}(s_0, p)$, we consider the reciprocal of the SINR function,

$$f(x) \;=\; \text{SINR}^{-1}(s_0, q) \;=\; \sum_{i=1}^{n-1} \left[\frac{\psi_i}{\psi_0} \left(\frac{x}{a_i + x} \right)^{\alpha} \right] + \frac{x^{\alpha}}{\psi_0} \cdot N \;,$$

and prove that $f(x) < f(1)$ for all $x \in (0, 1)$. This follows since the derivative $\frac{df(x)}{dx} = \frac{\alpha x}{\psi_0} \cdot \left(\sum_{i=1}^{n} \frac{\psi_i \cdot a_i}{(a_i + x)^{(\alpha+1)}} + N \right)$ is positive for $x \in (0, 1]$. ▮

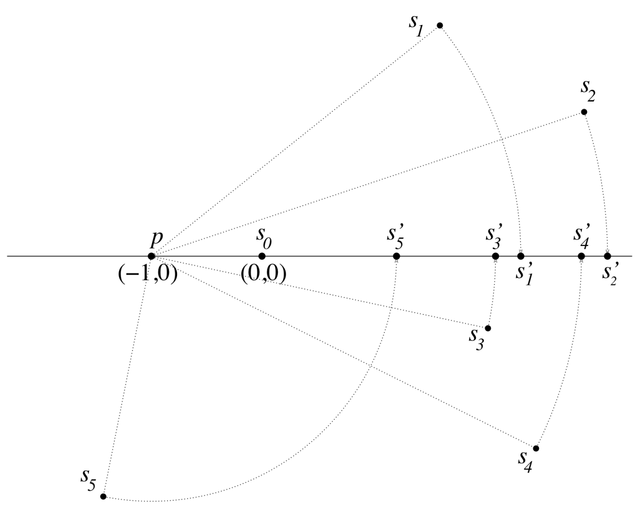

Figure 3: Relocating stations. All stations are mapped to the positive x-axis, so that the SINR value at point p with respect to the station s_0, is preserved.

2.1 Convexity without background noise

We now complete the proof for the noise free case where $N = 0$.

Lemma 2.4 *For every wireless network $\mathcal{A}_0 = \langle d, S, \bar{\psi}, N = 0, \beta, \alpha \rangle$, $\mathcal{VH}_{\mathcal{A}}(s_i)$ is convex for every $s_i \in S$.*

Proof: By Lemma 2.3, it remains to show that $\overline{p_1\, p_2} \subseteq \mathcal{VH}_{\mathcal{A}_0}(s_0)$ for any every points $p_1, p_2 \in \mathcal{VH}_{\mathcal{A}_0}(s_0)$ such that $s_0 \notin \overline{p_1\, p_2}$. Note that by the convexity of a Voronoi cell, $\overline{p_1\, p_2} \subset \text{Vor}(s_i)$. Thus, there is no station s_i on this segment, concluding that the $\text{SINR}_{\mathcal{A}_0}(s_0, p)$ function is continuous on the $\overline{p_1\, p_2}$ segment. It remains to prove that $\overline{p_1\, p_2} \subseteq \mathcal{H}_{\mathcal{A}_0}(s_0)$, i.e., that $\text{SINR}_{\mathcal{A}_0}(s_0, q) \geq \beta$ for any $q \in \overline{p_1\, p_2}$. We now show that for every $q \in \overline{p_1\, p_2}$,

$$\text{SINR}_{\mathcal{A}_0}(s_0, q) \geq \min\{\text{SINR}_{\mathcal{A}_0}(s_0, p_1), \text{SINR}_{\mathcal{A}_0}(s_0, p_2)\}.$$

Specifically, we show that the dual statement holds, namely, that

$$\mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0,q) \le \max\left\{\mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0,p_1), \mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0,p_2)\right\}. \tag{3}$$

By Lemma 1.1 and by the continuity of the $\mathrm{SINR}_{\mathcal{A}}$ function in the segment $\overline{p_1\,p_2}$, it is sufficient to consider the case where $p_1 = (-1,0)$, $p_2 = (1,0)$ and $q = (0,0)$, the middle point between p_1 and p_2 on the segment. By applying Lemma 2.2, we have

$$\max\{\mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0,p_1)\ ,\ \mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0,p_2)\} \ \ge\ \sum_{i=1}^{n-1} \frac{\psi_i}{\psi_0} \cdot \left(\frac{\rho_0+1}{\rho_i+1}\right)^{\alpha/2} . \tag{4}$$

On the other hand, by Eq. (2),

$$\mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0,q) \ =\ \sum_{i=1}^{n-1} \frac{\psi_i}{\psi_0} \cdot \left(\frac{\rho_0}{\rho_i}\right)^{\alpha/2} . \tag{5}$$

As $q \in \mathrm{V{\scriptsize OR}}(s_0)$, we have that $\rho_i > \rho_0$ and hence $\rho_0/\rho_i < (\rho_0+1)/(\rho_i+1)$ for every $i \in \{1,...,n-1\}$. This, together with Eq. (4) and (5), implies Ineq. (3). ▮

2.2 Convexity with background noise

We now consider the general case where $N \ge 0$.

Proof: [Theorem 2.1] Consider two points $p_1, p_2 \in \mathcal{VH}_{\mathcal{A}}(s_0)$. We need to show that $\overline{p_1\,p_2} \subseteq \mathcal{VH}_{\mathcal{A}}(s_0)$. By Lemma 1.1, we may assume without loss of generality that $p_1 = (-1,0)$ and $p_2 = (1,0)$. Let $d_N = \max\{\mathrm{dist}(s_0,p_1), \mathrm{dist}(s_0,p_2)\}$.

Let $\mathcal{A}^*$ be a noise-free $(n+1)$-station network obtained from $\mathcal{A}$ by replacing the background noise with a new station s_N located in $(0,d_N)$ with transmission power $\psi_N = N \cdot (d_N^2+1)^{\alpha/2}$. That is, $\mathcal{A}^* = \langle d=2, S^*, \bar{\psi}^*, N=0, \beta, \alpha\rangle$, where $S^* = S \cup \{s_N\}$ and $\bar{\psi}^* = (\psi_0, ..., \psi_{n-1}, \psi_N)$. It is easy to verify that $\psi_N \cdot \mathrm{dist}(s_N,p_i)^{-\alpha} = N$ and $\psi_N \cdot \mathrm{dist}(s_N,q)^{-\alpha} \ge N$, for every $q \in \overline{p_1\,p_2}$. Thus, on the one hand,

$$\mathrm{SINR}_{\mathcal{A}^*}(s_0,p_i) = \mathrm{SINR}_{\mathcal{A}}(s_0,p_i), \text{ for } i \in \{1,2\}, \tag{6}$$

and on the other hand, for all points $q \in \overline{p_1\,p_2}$,

$$\mathrm{SINR}_{\mathcal{A}}(s_0,q) \ge \mathrm{SINR}_{\mathcal{A}^*}(s_0,q). \tag{7}$$

We now show that $p_1,p_2 \in \mathcal{VH}_{\mathcal{A}^*}(s_0)$. We first claim that $p_1,p_2 \in \mathrm{V{\scriptsize OR}}^*(s_0)$ where $\mathrm{V{\scriptsize OR}}^*$ is the Voronoi diagram of the set S^*. Since $p_1,p_2 \in \mathcal{VH}_{\mathcal{A}}(s_0)$, in particular $p_1,p_2 \in \mathrm{V{\scriptsize OR}}(s_0)$. This implies that $\mathrm{dist}(s_0,p_i) \le \mathrm{dist}(s_j,p_i)$, for every $i \in \{1,2\}$ and $j \in \{1,...,n-1\}$. In addition, $\mathrm{dist}(s_N,p_i) > d_N \ge \mathrm{dist}(s_0,p_i)$, implying that $p_1,p_2 \in \mathrm{V{\scriptsize OR}}^*(s_0)$ as needed. It remains to show that $p_1,p_2 \in \mathcal{H}_{\mathcal{A}^*}(s_0)$. Since $p_1,p_2 \in \mathcal{H}_{\mathcal{A}}(s_0)$, $\mathrm{SINR}_{\mathcal{A}}(s_0,p_i) \ge \beta$ for $i \in \{1,2\}$. Thus, by Eq. (6), $\mathrm{SINR}_{\mathcal{A}^*}(s_0,p_i) \ge \beta$ as well, and $p_1,p_2 \in \mathcal{H}_{\mathcal{A}^*}(s_0)$. Finally, since $p_1,p_2 \in \mathcal{VH}_{\mathcal{A}^*}(s_0)$ where $\mathcal{A}^*$ is a noise free network, by Lemma 2.4 it holds that $\mathrm{SINR}_{\mathcal{A}^*}(s_0,q) \ge \beta$, for all points $q \in \overline{p_1\,p_2}$. Thus, by Ineq. (7), also $\mathrm{SINR}_{\mathcal{A}}(s_0,q) \ge \beta$, for all points $q \in \overline{p_1\,p_2}$, are required. The lemma follows. ▮

Theorem 2.1 is established.

3 Fatness of SINR+Voronoi Zones

In this section we develop a deeper understanding of the shape of SINR+Voronoi reception zones by analyzing their fatness. Consider a nonuniform power network $\mathcal{A} = \langle d, S, \bar{\Psi}, N, \beta, \alpha \rangle$ with positive background noise $N > 0$, where $S = \{s_0, \ldots, s_{n-1}\}$, and $\alpha \geq 0$ and $\beta > 1$ are constants[3].

We focus on s_0 and assume that its location is not shared by any other station (otherwise, $\mathcal{H}(s_0) = \{s_0\}$). Let $\kappa = \min_{s_i \in S \setminus \{s_0\}} \{\text{dist}(s_0, s_i)\}$ denote the distance between s_0 and the closest interfering station. The following fact summarizes the known fatness bounds for uniform and nonuniform reception zones.

Fact 3.1 *Let $\mathcal{A}_u$ (resp., $\mathcal{A}_{nu}$) be an n-station uniform (resp., nonuniform) network. Then*
(a) $\varphi(s_0, \mathcal{H}_{\mathcal{A}_\mathsf{u}}(s_0)) = O(1)$, and
(b) $\varphi(s_0, \mathcal{H}_{\mathcal{A}_\mathsf{nu}}(s_0)) = O(\psi_{\max}/\kappa \cdot \sqrt{n/N})$ for $\alpha = 2$.

We now show that in the SINR+Voronoi setting, the fatness of $\mathcal{VH}_{\mathcal{A}}(s_0)$ with respect to s_0, can be bounded as a function of $\psi_{\max}$, κ, α, β and N, namely, it is independent of the number of stations n.

Theorem 3.2

$$\varphi(s_0, \mathcal{VH}(s_0)) \;\leq\; \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1} \cdot \max\left\{1 \;\;,\;\; \frac{3}{\kappa} \cdot \sqrt[\alpha]{\frac{\psi_0}{N \cdot \beta}} \cdot \max\{1, \sqrt[\alpha]{\beta}-1\}\right\} \;.$$

In certain cases, tighter bounds can be obtained. An SINR+Voronoi zone $\mathcal{VH}_{\mathcal{A}}(s_0)$ is *well-bounded* if the minimal enclosing ball of $\mathcal{VH}_{\mathcal{A}}(s_0)$ is fully contained in the Voronoi cell $\textsc{Vor}(s_0)$. We next claim that the fatness of well-bounded zones is constant.

Lemma 3.3 *Let $\mathcal{VH}_{\mathcal{A}}(s_0)$ be a well-bounded zone, then $\varphi(s_0, \mathcal{VH}_{\mathcal{A}}(s_0)) = O(1)$.*

The proof of Thm. 3.2 is provided in Appendix A. Its overall structure is similar to that of Thm. 4.2 in [3], but requires delicate adaptations for the nonuniform setting. Bounding the radius $\Delta(s_0, \mathcal{VH}_{\mathcal{A}}(s_0))$ is easily obtained by considering the extreme case where s_0 is the solitary transmitting stations. Our main efforts went into bounding the small radius $\delta(s_0, \mathcal{VH}_{\mathcal{A}}(s_0))$ as a function that is independent in n. The proof consists of three main steps. First (in Subsec. A.1) we bound the fatness of SINR+Voronoi zones in a setting of two stations in a one-dimensional space. Then (in Subsec. A.2), we consider a special type of nonuniform power networks called *positive collinear* networks. Finally (in Subsec. A.3), the general case is reduced to the case of positive collinear networks.

4 Applications

In this section, we present two applications for the properties established in the previous sections. In Subsec. 4.1, we present an application for the convexity property and describe a new variant of the power control problem. In Subsec. 4.2, we exploit the convexity and the improved bound on the fatness of SINR+Voronoi zones to obtain an improved approximate point location scheme for SINR+Voronoi diagram.

[3]Note that the convexity proof presented in Section 2 holds for any $\beta \geq 0$.

4.1 The Power Control Voronoi Diagram (PCVD) Problem

In the standard power control problem for wireless networks, one is given a set of n communication links $L = \{\ell_0, \ldots, \ell_{n-1}\}$, where each link ℓ_i represents a communication request from station s_i to receiver r_i. The question is then to find an optimal power assignment for the stations, so as to make the reception threshold β as high as possible and ease the decoding process. As it turns out, this problem can be solved elegantly using the Perron–Frobenius (PF) Theorem [18]. Essentially, since every station is required to satisfy a fixed number of receivers (in the standard formulation, there is actually one receiver per station), the system can be represented in matrix form that has some useful properties.

We now consider a new variant of the problem in which every station has to satisfy a continuous *zone* rather than a fixed number of points. The motivation for this formulation is that it allows one to attain an optimal complete coverage of the reception map. We now define the problem formally.

In the *Power Control for Voronoi Diagram* (PCVD) problem, one is given a network of n stations $S = \{s_0, \ldots, s_{n-1}\}$ embedded in some d-dimensional bounded polygon[4] $\mathcal{P}$ and the task is to find an optimal power assignment for the stations, so as to make the reception threshold β as high as possible such that $\mathrm{SINR}_{\mathcal{A}}(s_i, p) \geq \beta$ for every s_i and every point $p \in \textsc{Vor}(s_i) \cap \mathcal{P}$.

Note that without the convexity property within $\mathcal{VH}_{\mathcal{A}}(s_i)$ zones, established in the previous section, it is unclear how to formulate this problem by using a *finite* set of inequalities. This is because each Voronoi cell consists of infinitely many reception points, each of which must satisfy an SINR constraint. Due to the convexity property, we can provide the following succinct representation of the problem. For every station $s_i \in S$, let $\mathcal{V}_i$ be the vertex set[5] of the bounded polytope $\textsc{Vor}(s_i) \cap \mathcal{P}$. Let $m = \sum_{i=0}^{n-1} |\mathcal{V}_i|$. The optimization task consists of m inequalities and $n+1$ variables (n variables corresponding to the power assignment and β). This yields the following formulation.

$$\begin{aligned} &\text{maximize } \beta \text{ subject to:} \\ &\mathrm{SINR}(s_i, p) \geq \beta \text{ for every } s_i \in S \text{ and } p \in \mathcal{V}_i \ . \end{aligned} \tag{8}$$

We first claim that this is a correct formulation for the Power Control for Voronoi Diagram problem. Let β^* be the optimum solution of Program (8). By the feasibility of this solution, $\mathrm{SINR}(s_i, p) \geq \beta^*$ for every $p \in \mathcal{V}_i$. Since the reception zone is convex within its Voronoi cell, we get that $\mathrm{SINR}(s_i, p) \geq \beta^*$ for every $p \in \textsc{Vor}(s_i)$ (in particular, in the optimum β, the reception zone contains the Voronoi cell of the station).

To solve Program (8), note that for any fixed β, the inequalities are linear in the n transmission power variables and hence the resulting set of m linear inequalities is solvable in polynomial time. A nearly optimum power assignment can then be found by searching for the best β via binary search up to some desired approximation.

4.2 The Closest Station Point Location Problem

In the *Closest Station Point Location Problem*, one is given a nonuniform power network $\mathcal{A}$ with n transmitting stations, $S = \{s_0, \ldots, s_{n-1}\}$. Given a query point $p \in \mathbb{R}^2$, it is required to answer whether s_p is heard at p where s_p is the closest station to p (i.e., $p \in \textsc{Vor}(s_p)$).

[4]the role of $\mathcal{P}$ is to guarantee that all Voronoi cells restricted to $\mathcal{P}$ are bounded.

[5]Note that the $\mathcal{V}_i$ sets are not disjoint and hence vertices are counted multiple times

Since nonuniform SINR zones are not convex and non-fat, the preprocessing as well as the memory required in the approximate point location scheme of [10] are polynomial but costly. In this section, we show that one can solve approximate point location tasks for *nonuniform* networks with the effectively the same bounds as obtained for *uniform* (i.e., in case where ψ_{max} and the N are bounded by some constant) as long as the query point p belongs to the Voronoi cell of the target station that should be heard at p. By Lemma 5.1 of [3], we have the following.

Theorem 4.1 *For every n-station nonuniform power network with SINR+Voronoi reception zones $\langle \mathcal{VH}_{\mathcal{A}}(s_1), \ldots, \mathcal{VH}_{\mathcal{A}}(s_n) \rangle$, it is possible to construct, in $O((\psi_{\max}/(\kappa \cdot N))^{3/\alpha} \cdot n^2 \cdot \epsilon^{-1})$ preprocessing time, a data structure* `DS` *requiring memory of size $O((\psi_{\max}/(\kappa \cdot N))^{3/\alpha} \cdot n \cdot \epsilon^{-1})$ that imposes a $(2n+1)$-wise partition $\widetilde{\mathcal{VH}} = \langle \mathcal{VH}^{+}_{\mathcal{A}}(s_1), \ldots, \mathcal{VH}^{+}_{\mathcal{A}}(s_n), \ldots, \mathcal{VH}^{?}_{\mathcal{A}}(s_1), \ldots, ReceptionZoneVor^{?}_{\mathcal{A}}(s_n), \mathcal{VH}^{-}_{\mathcal{A}} \rangle$. of the Euclidean plane, such that for every $i \in \{0, \ldots, n-1\}$*

(a) $\mathcal{VH}^{+}_{\mathcal{A}}(s_i) \subseteq \mathcal{VH}_{\mathcal{A}}(s_i)$.

(b) $\mathcal{VH}_{\mathcal{A}}(s_i) \cap \mathcal{VH}^{-}_{\mathcal{A}} = \emptyset$.

(c) $\mathcal{VH}^{?}_{\mathcal{A}}(s_i)$ *is bounded and its area is at most an ϵ-fraction of the area of $\mathcal{VH}_{\mathcal{A}}(s_i)$.*

Furthermore, given a query point p, it is possible to extract from `DS` *in time $O(\log n)$, the zone in $\widetilde{\mathcal{VH}}$ to which p belongs, hence the closest station point location query can be answered with approximation ϵ with query time of $O(\log(\psi_{\max} \cdot n/(N \cdot \kappa)))$ where $\kappa = \min_{i,j} \operatorname{dist}(s_i, s_j)$.*

For comparison, the general point location scheme of [10] requires preprocessing time of $O(n^{10}\psi^4_{\max}/\epsilon^2)$ and memory of size $O(n^8\psi^4_{\max}/\epsilon^2)$.

5 Conclusion

The Voronoi diagram of the base stations is a natural model for wireless networks in the plane. In this paper, we show that the restriction of the nonuniform reception zone to the corresponding Voronoi region is as nice almost as uniform reception zones. The increasing demand for mobile networks and high performance networks has created a need to dynamically determine the power each base station should transmit in order to optimize the capacity of the network. A common approach is to assign each base station its own Voronoi cell. Once the network is dynamic, the Voronoi cell is no longer fixed and one can no longer compute, in advance, the parameters required for optimal network performance. We consider a fundamental problem, named as the Power Control for a Voronoi Diagram problem. The convexity property within Voronoi regions enables us to discretize the PCVD problem while maintaining optimality. In addition, we showed the point location queries for SINR+Voronoi zones can be solved with almost the same bounds as for the uniform case. We believe that this approach would pave the way for designing additional algorithms for dynamic mobile networks

References

[1] B. Aronov and M.J. Katz. Batched point location in SINR diagrams via algebraic tools. In *ICALP*, 2015.

[2] F. Aurenhammer and H. Edelsbrunner. An optimal algorithm for constructing the weighted voronoi diagram in the plane. *Pattern Recognition*, 17, 1984.

[3] C. Avin, Y. Emek, E. Kantor, Z. Lotker, D. Peleg, and L. Roditty. SINR diagrams: Convexity and its applications in wireless networks. *J. ACM*, 59(4), 2012.

[4] C. Avin, Z. Lotker, and Y.-A. Pignolet. On the power of uniform power: Capacity of wireless networks with bounded resources. In *Proc. of ESA*, pages 373–384, 2009.

[5] J. Basch, L.J. Guibas, and J. Hershberger. Data structures for mobile data, 1997.

[6] B. Chen, K. Jamieson, H. Balakrishnan, and R. Morris. Span: An energy-efficient coordination algorithm for topology maintenance in ad hoc wireless networks. *Wireless Networks*, 8:481–494, 2002.

[7] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer-Verlag, 2008.

[8] L.J. Guibas, J.S.B Mitchell, and T. Roos. Voronoi diagrams of moving points in the plane. In *Proc. of WG*. SV, 1992.

[9] E. Kantor, Z. Lotker, M. Parter, and D. Peleg. The topology of wireless communication. In *Proc. STOC*, 2011.

[10] E. Kantor, Z. Lotker, M. Parter, and D. Peleg. The topology of wireless communication. In *http://arxiv.org/pdf/1103.4566v2.pdf*, 2011.

[11] E. Kantor, Z. Lotker, M. Parter, and D. Peleg. The minimum principle of SINR and applications. Submitted, 2015.

[12] Y. Kim, J. Kim, H. Nam, and S. An. Hex-grid based routing protocol in wireless sensor networks. In *Computat. Sci. & Eng.*, pages 683–688, 2012.

[13] F. G. Nocetti, I. Stojmenovic, and J. Zhang. Addressing and routing in hexagonal networks with applications for tracking mobile users and connection rerouting in cellular networks. *IEEE Trans. Par. & Distr. Syst.*, 13:963–971, 2002.

[14] L.R. Ping, G. Rogers, S. Zhou, and J. Zic. Topology control with hexagonal tessellation. 2:91–98, 2007.

[15] I. Stojmenovic. Honeycomb networks: Topological properties and communication algorithms. *IEEE Trans. Par. & Distr. Syst.*, 8:1036–1042, 1997.

[16] X. Xu, Y. Li, R. Gao, and X. Tao. Joint voronoi diagram and game theory-based power control scheme for the hetnet small cell networks. *EURASIP J. Wireless Comm. and Networking*, 2014:213, 2014.

[17] S. Yousefi, M. S. Mousavi, and M. Fathy. Vehicular adhoc networks (vanets): challenges and perspectives. In *IEEE ITS Telecomm.*, pages 761–766, 2006.

[18] J. Zander. Performance of optimum transmitter power control in cellular radiosystems. *IEEE Tr. Vehic. Technol.*, 41:57–62, 1992.

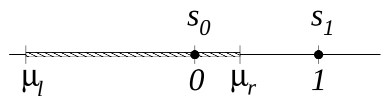

Figure 4: The embedding of s_0 and s_1 in a one-dimensional space.

APPENDIX

A Fatness

A.1 Two stations in a one-dimensional space (without noise)

Let $\mathcal{A} = \langle 1, \{s_0, s_1\}, (\psi_0, \psi_1), N = 0, \beta > 1, \alpha \geq 1\rangle$ be a nonuniform network consisting of two stations s_0, s_1 embedded in the one-dimensional space $\mathbb{R}$ with no background noise (i.e., $N = 0$). By Lemma 1.1, we can assume without loss of generality that s_0 is located at $a_0 = 0$ and s_1 is located at $a_1 = \kappa > 0$. Let $\mu_r = \min\{\kappa/2, \max\{p > 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0, p) \geq \beta\}\}$ and let $\mu_l = \min\{p < 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0, p) \geq \beta\}$ (see Figure 4), if $\psi_0/\psi_1 \geq \beta$ and $\mu_l = -\infty$, otherwise. It is easy to verify that $\mathcal{H}(s_0) = [\mu_l, \mu_r]$ if $\psi_0/\psi_1 \geq \beta$ and $\mathcal{H}_0 = (-\infty, \mu_r]$, otherwise. Thus, $\delta = \delta(s_0, \mathcal{VH}(s_0)) = \mu_r$ and $\Delta = \Delta(s_0, \mathcal{VH}(s_0)) = -\mu_l$.

Lemma A.1 *The network $\mathcal{A}$ satisfies the following:*

1. $\delta(s_0, \mathcal{VH}(s_0)) = \min\{\kappa/2, \frac{\kappa}{1+\sqrt[\alpha]{\beta\psi_1/\psi_0}}\}$,
2. *If $\psi_1 \geq \psi_0$, then $\Delta(s_0, \mathcal{H}(s_0)) = \frac{\kappa}{1-1\sqrt[\alpha]{\beta\psi_1/\psi_0}}$, and $\varphi(s_0, \mathcal{VH}(s_0)) = \Delta/\delta \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$, with equality when $\psi_1 = \psi_0$.*

Proof: Let begin by showing assertion (1) of the lemma. Let $(x, 0)$ for $x > 0$ be the boundary point of $\mathcal{H}(s_0)$ on the x-axis, i.e., satisfying the linear equation

$$\frac{\psi_0/x^\alpha}{\psi_1/(\kappa - x)^\alpha} = \beta,$$

leading to $\frac{\kappa-x}{x} = (\beta\psi_1/\psi_0)^{1/\alpha}$, or, $x + (\beta\psi_1/\psi_0)^{1/\alpha}x = \kappa$. Solving this linear equation for positive x yields,

$$\mu_r \;=\; \min\{\frac{\kappa}{2}, \frac{\kappa}{1 + \sqrt[\alpha]{\beta\psi_1/\psi_0}}\} \ ,$$

as needed for assertion (1).

Now, we prove that assertion (2) holds. So, suppose that $\psi_1 \geq \psi_0$. In this case, by part (1), $\mu_r = \kappa/(1 + \sqrt[\alpha]{\beta\psi_1/\psi_0})$. Similarly to the boundary point μ_r, the boundary point μ_l of $\mathcal{H}(s_0)$ is obtained by solving the equation $\frac{\psi_0/(-x)^\alpha}{\psi_1/(\kappa-x)^\alpha} = \beta$, for negative x, yielding

$$\mu_l \;=\; \frac{\kappa}{1 - \sqrt[\alpha]{\beta\psi_1/\psi_0}} \ .$$

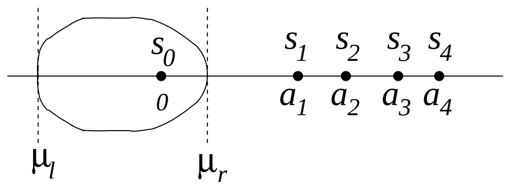

Figure 5: A positive collinear network.

Therefore the ratio Δ/δ satisfies

$$\frac{\Delta}{\delta} \;=\; \frac{-\mu_l}{\mu_r} \;=\; \frac{\sqrt[\alpha]{\beta\psi_1/\psi_0}+1}{\sqrt[\alpha]{\beta\psi_1/\psi_0}-1} \;\leq\; \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}\,,$$

where the last inequality holds since $\alpha > 0$, $\beta > 1$ and $\psi_1/\psi_0 \geq 1$. The Lemma follows. ▮

A.2 Positive collinear networks

In this section we turn to the Euclidean plane $\mathbb{R}^2$ and consider a special type of nonuniform power networks. A network $\mathcal{A} = \langle 2, \{s_0, \ldots, s_{n-1}\}, \bar{\Psi}, N, \beta, \alpha\rangle$ is said to be *positive collinear* if $s_0 = (0,0)$ and $s_i = (a_i, 0)$ for some $a_i > 0$ for every $1 \leq i \leq n-1$. Positive collinear networks play an important role in the subsequent analysis due to the following lemma. See Figure 5 for an illustration.

Lemma A.2 *Let $\mathcal{A}$ be a positive collinear nonuniform power network with positive background noise $N > 0$. Fix $\kappa = \min_{i\in\{1,\ldots,n-1\}} \mathrm{dist}(s_0, s_i)$, $\mu_r = \min\{\kappa/2, \max\{r > 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0,(r,0)) \geq \beta\}\}$ and $\mu_l = \min\{r < 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0,(r,0)) \geq \beta\}$. Then*

1. *$\delta(s_0, \mathcal{VH}(s_0)) = \mu_r$,*
2. *$\Delta(s_0, \mathcal{VH}(s_0)) = -\mu_l$,*
3. *$\varphi(s_0, \mathcal{VH}(s_0)) = -\frac{\mu_l}{\mu_r} \leq \max\left\{\frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1},\ \sqrt[\alpha]{\frac{\psi_0}{N\cdot\beta}}\cdot\frac{\sqrt[\alpha]{\beta}+1}{\kappa}\right\}$, and*
4. *if $\Delta(s_0, \mathcal{VH}(s_0)) \geq \kappa$, then $\delta(s_0, \mathcal{VH}(s_0)) \geq \frac{\kappa}{\sqrt[\alpha]{\beta}+1}\cdot\min\left\{\sqrt[\alpha]{\beta}-1\ ,\ 1\right\}$.*

Proof: First, we argue that the SINR+Voronoi zone $\mathcal{VH}(s_0)$ of s_0 in the network $\mathcal{A}$ is contained in the infinite vertical strip defined by $\mu_l \leq x \leq \mu_r$. To see why this is true, suppose, towards contradiction, that the point $(x,y) \in \mathcal{VH}(s_0)$ for some $x > \mu_r$ or $x < \mu_l$. By symmetry considerations, the point $(x,-y)$ is also in $\mathcal{VH}(s_0)$. By the convexity of $\mathcal{VH}(s_0)$, it follows that $(x,0) \in \mathcal{VH}(s_0)$, in contradiction to the definitions of μ_r and μ_l. We thus have the following.

Claim A.3 *If $(x,y) \in \mathcal{VH}(s_0)$, then $\mu_l \leq x \leq \mu_r$.*

To prove assertion (1) of the lemma, we show that the ball of radius μ_r centered at s_0 is contained in $\mathcal{VH}(s_0)$. In fact, by the convexity of $\mathcal{VH}(s_0)$, it is sufficient to show that the point $p(\theta) = (\mu_r\cos\theta, \mu_r\sin\theta)$ is in $\mathcal{VH}(s_0)$ for all $0 \leq \theta \leq \pi$. Since the network is positive collinear, it follows that $\mathrm{I}_{\mathcal{A}}(s_0, p(\theta))$ attains its maximum for $\theta = 0$. Therefore the fact that $p(0) = (\mu_r, 0) \in$

$\mathcal{H}(s_0)$ implies that $p(\theta) \in \mathcal{VH}(s_0)$ for all $0 \le \theta \le \pi$ as desired. Assertion (1) follows. Next, we show that Δ is realized by the point $(\mu_l, 0)$. Indeed, by the triangle inequality, all points at distance k from s_0 are at distance at most $k + a_i$ from $s_i = (a_i, 0)$, with equality attained for the point $(-k, 0)$. Thus the minimum interference to s_0 under $\mathcal{A}$ among all points at distance k from s_0 is attained at the point $(-k, 0)$. Therefore, by the definition of μ_l, there cannot exist any point $p \in \mathcal{VH}(s_0)$ such that $\text{dist}(p, s_0) > -\mu_l$. Assertion (2) follows.

It remains to establish assertions (3) and (4). Recall that the leftmost station other than s_0 is located at $(\kappa, 0)$. By definition, $\mu_r < \kappa/2$. Denote the energy of station s_i at $(\mu_r, 0)$ by $\mathcal{E}_i = \text{E}(s_i, (\mu_r, 0)) = \psi_i \cdot (a_i - \mu_r)^{-\alpha}$. We construct a new $(n+1)$-station network $\mathcal{A}' = \langle 2, S', \psi', 0, \beta, \alpha \rangle$ consisting of s_0 and n new stations $s'_0, \ldots, s'_{n-1}$, all located at $(\kappa, 0)$. We set the transmission power ψ'_i of the new stations s'_i to

$$\psi'_i = \begin{cases} \mathcal{E}_i \cdot (\kappa - \mu_r)^\alpha & \text{for } 1 \le i \le n-1; \text{ and} \\ N \cdot (\kappa - \mu_r)^\alpha & \text{for } i = n. \end{cases}$$

This ensures that the energy produced by these stations at $(\mu_r, 0)$ is

$$\text{E}(s'_i, (\mu_r, 0)) = \begin{cases} \mathcal{E}_i & \text{for } 1 \le i \le n-1, \text{ and} \\ N & \text{for } i = n. \end{cases}$$

Let $\Delta = \Delta(s_0, \mathcal{VH}_{\mathcal{A}}(s_0))$, $\Delta' = \Delta(s_0, \mathcal{VH}_{\mathcal{A}'}(s'_0))$. The small radii δ and δ' are defined analogously. Note that the Voronoi cell of s_0 is the same in both networks $\mathcal{A}$ and $\mathcal{A}'$.

The network $\mathcal{A}'$ falls into the setting of Subsec. A.1: the stations $s'_1, \ldots, s'_n$ share the same location, thus they can be considered as a single station $\hat{s}_1$ with transmission power $\hat{\psi}_1 = \sum_{i=1}^n \psi'_i$.

Define $\mu'_r = \min\{\kappa/2, \max\{r > 0 \mid \text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \ge \beta\}\}$

$$\mu'_l = \begin{cases} \min\{r < 0 \mid \text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \ge \beta\} & \text{if } \hat{\psi}_1 \ge \beta\psi_0 \\ -\infty & \text{otherwise.} \end{cases}$$

The restriction of the $\mathcal{VH}_{\mathcal{A}'}(s_0)$ to the x-axis is thus $[\mu'_l, \mu'_r]$. In addition, it is easy to verify that $\Delta' \le \sqrt[\alpha]{\psi_0 / N \cdot \beta}$ (i.e., this is attained when only s_0 transmits). By A.1, $-\mu'_l/\mu'_r \le \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$, if $\hat{\psi}_1 \ge \psi_0$ and $\delta' \le \frac{\kappa}{1+\sqrt[\alpha]{\beta}}$ otherwise.

The remaining of the proof relies on establishing the following two bounds.

(A1) $\text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \le \text{SINR}_{\mathcal{A}}(s_0, (r, 0))$ for all $\mu_r \le r < \kappa$; and

(A2) $\text{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \ge \text{SINR}_{\mathcal{A}}(s_0, (r, 0))$ for all $r \le \mu_r$, $r \ne 0$.

By combining bounds (A1) and (A2), we conclude that $\mu'_r \le \mu_r$ and $\mu'_l \le \mu_l$. Thus, $\Delta' \ge \Delta$ and $\delta' \le \delta$. Assertion (3) of the lemma holds, by combining this together with the facts that (i) $\Delta/\delta \le \Delta'/\delta' \le \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$, if $\hat{\psi}_1 \ge \psi_0$, and (ii) $\delta \ge \kappa/(1 + \sqrt[\alpha]{\beta})$ and $\Delta \le \psi_0/(\beta \cdot N)$, otherwise.

For showing assertion (4), we consider two cases. If $\hat{\psi}_1 < \psi_0$, then as mention above, we have that $\delta \ge \delta' \ge \frac{\kappa}{\sqrt[\alpha]{\beta}+1}$ as needed for assertion (4). Otherwise, suppose that $\hat{\psi}_1 \ge \psi_0$ and that $\Delta \ge \kappa$ (the second inequality is the condition of that assertion). Combining this with the facts that $\Delta' \ge \Delta$, $\delta' \le \delta$ and with Assertion (2) of Lemma A.1,

$$\kappa/\delta \le \Delta/\delta \le \Delta'/\delta' \le \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1} ,$$

which completes the proof of Assertion (4).

To establish Inequalities (A1) and (A2), consider some point $p = (r, 0)$, where $r < \kappa$, $r \neq 0$. For every $1 \leq i \leq n-1$, we have

$$\mathrm{E}(s_i, p) = \psi_i \cdot (a_i - r)^{-\alpha} \ , \quad \text{while} \quad \mathrm{E}(s_i', p) = \psi_i' \cdot (\kappa - r)^{-\alpha} = \frac{\psi_i \cdot (\kappa - \mu_r)^\alpha}{(\kappa - r)^\alpha (a_i - \mu_r)^\alpha} \ .$$

Comparing these two expressions, we get $\mathrm{E}(s_i, p) \geq \mathrm{E}(s_i', p)$, or equivalently, $(\kappa - r)(a_i - \mu_r) \geq (\kappa - \mu_r)(a_i - r)$. Rearranging, $\kappa a_i - \kappa\mu_r - a_i r + r\mu_r \geq \kappa a_i - \kappa r - a_i\mu_r + r\mu_r$, or

$$\mu_r(a_i - \kappa) \ \geq \ r(a_i - \kappa) \ ,$$

Note that the inequality holds with equality if and only $a_i = \kappa$, which, by definition, implies that $\mathrm{E}(s_i, p) = \mathrm{E}(s_i', p)$. Therefore, the contribution of s_i' to the total interference at $p = (0, r)$ is not larger than that of s_i as long as $r \leq \mu_r$ and not smaller than that of s_i as long as $\mu_r \leq r < \kappa$. On the other hand, the energy of s_n' at $p = (r, 0)$ satisfies $\mathrm{E}(s_n', p) \leq N$ for all $\kappa \leq \mu_r$ and $\mathrm{E}(s_n', p) \geq N$ for all $\mu_r \leq r < \kappa$. Inequalities (A1) and (A2) follow. ▮

A.3 General uniform power networks in d-dimensional space

We are now ready to prove Thm. 3.2.

Proof: [Proof of Theorem 3.2.] Consider an arbitrary nonuniform power network $\mathcal{A} = \langle d, S, \bar{\Psi}, N > 0, \beta, \alpha \rangle$, with positive noise, where $S = \{s_0, \ldots, s_{n-1}\}$ and $\beta > 1$ is a constant. We employ Lemma 1.1 to assume without loss of generality that s_0 is located at $(0, ..., 0)$ and that $\max\{\mathrm{dist}(s_0, q') \mid q' \in \mathcal{VH}_{\mathcal{A}}(s_0)$ is realized by a point $q' = (-\Delta, 0, ..., 0)$ on the negative x-axis. Let

$$q = \begin{cases} (-\Delta, 0, ..., 0) & \text{if } \Delta \leq \kappa/3 \\ (-\kappa/3, 0, ..., 0). & \end{cases}$$

Note that by definition, q is an internal point in $\mathrm{Vor}(s_0)$. By the convexity of the $\mathcal{VH}_{\mathcal{A}}(s_0)$, it holds that $q \in \mathcal{VH}_{\mathcal{A}}(s_0)$. We now construct a new positive collinear nonuniform power network $\mathcal{A}' = \langle d, \{s_0, s_1', \ldots, s_{n-1}'\}, \bar{\Psi}, N > 0, \beta, \alpha \rangle$, obtained from $\mathcal{A}$ by rotating each station s_i around the point q until it reaches the Euclidean plane at the positive x-axis (see Figure 6). More formally, the location of s_0 remains unchanged and $s_i' = (a_i', 0)$, where $a_i' = \mathrm{dist}(s_i, q) - \mathrm{dist}(s_0, q)$ for every $1 \leq i \leq n-1$. Note that, since $q \in \mathrm{Vor}(s_0)$, it holds that $\mathrm{dist}(s_0, q) < \mathrm{dist}(s_i, q)$ (and hence $a_i' > 0$), for every $i = 1, ..., n-1$. Thus, $\mathcal{A}'$ is a positive collinear network. In addition, for every $i = 1, ..., n-1$, the following three properties hold.
(P1) $\mathrm{dist}(s_i', q) = \mathrm{dist}(s_i, q)$;
(P2) $\mathrm{dist}(s_0', s_i') \leq \mathrm{dist}(s_0, s_i)$; and
(P3) $\mathrm{dist}(s_0', s_i') \geq \kappa/3$.
Property (P1) is trivially holds by the contraction of $\mathcal{A}'$ in which the stations preserves the distance from q. Property (P2) holds, since $\mathrm{dist}(s_0', s_i') = \mathrm{dist}(s_i, q) - \mathrm{dist}(s_0, q)$, whereas $\mathrm{dist}(s_0, s_i) \geq \mathrm{dist}(s_i, q) - \mathrm{dist}(s_0, q)$. Finally, Property (P3) holds, since (i) $\mathrm{dist}(s_0, q) \leq \kappa/3$; (ii) $\mathrm{dist}(s_0, s_i) \geq \kappa$, hence, $(iii)\mathrm{dist}(s_i, q) \geq 2\kappa/3$. Property (P3) holds, by combining together (ii), (iii) with (P1).

Let $\kappa' = \min_{i \in \{1, \ldots, n-1\}} \mathrm{dist}(s_0, s_i')$ be the distance from s_0 to the closest interfering station in $\mathcal{A}'$. By Property (P2), it follows that $\kappa' \leq \kappa$ and by Property (P3) it follows that $\kappa' \geq \kappa/3$. We now consider the fatness of $\mathcal{VH}_{\mathcal{A}'}(s_0)$. Let $\delta' = \max\{r > 0 \mid B(s_0, r) \subseteq \mathcal{VH}_{\mathcal{A}'}(s_0)\}$ and

$\Delta' = \min\{r > 0 \mid B(s_0, r) \supseteq \mathcal{VH}_{\mathcal{A}'}(s_0)\}$ be the radii of the maximum (resp., minimum) balls centered at s_0 that bound $\mathcal{VH}_{\mathcal{A}'}(s_0)$. Let $\mu'_r = \min\{\kappa'/2, \max\{r > 0 \mid \mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0, ..., 0)) \geq \beta\}\}$ and let $\mu'_l = \min\{r < 0 \mid \mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0, ..., 0)) \geq \beta\}$ be the extreme points on the x-axis that belong to $\mathcal{VH}_{\mathcal{A}'}(s_0)$. By Lemma A.2, Assertion 3, $\delta' = \mu'_r$ and $\Delta' = -\mu'_l$ and

$$\varphi(s_0, \mathcal{VH}_{\mathcal{A}'}(s_0)) = -\frac{\Delta'}{\delta'} \leq \max\left\{\frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}, \sqrt[\alpha]{\frac{\psi_0}{N \cdot \beta}} \cdot \frac{\sqrt[\alpha]{\beta}+1}{\kappa'}\right\} \tag{A.1}$$

and by Assertion (4) of that lemma, if $\Delta' \geq \kappa'$, then

$$\delta' \geq \frac{\kappa'}{\sqrt[\alpha]{\beta}+1} \cdot \min\{1\ ,\ \ \sqrt[\alpha]{\beta}-1\}\ . \tag{A.2}$$

The remaining of the proof relies on establishing that

$$\delta \geq \delta' \tag{A.3}$$

and that

$$\Delta = \Delta',\ \text{if } \Delta \leq \kappa/3. \tag{A.4}$$

We now consider the following two cases.

Case 1: $\Delta \geq \kappa/3$. By Inequality (A.2) and Inequality (A.3) together with that fact that $\kappa' \geq \kappa/3$ we have, on the one hand, that

$$\delta \geq \frac{\kappa}{3(\sqrt[\alpha]{\beta}+1)} \cdot \min\{1\ ,\ \ \sqrt[\alpha]{\beta}-1\}\ .$$

On the other hand, clearly,

$$\Delta \leq \sqrt[\alpha]{\frac{\psi_0}{N \cdot \beta}}$$

(the above inequality is attained with equality when only s_0 transmits). Hence, the theorem follows.

Case 2: $\Delta < \kappa/3$. The theorem follows by combining together inequalities (A.1) and (A.3) with Equality (A.4) with the fact that $\kappa' \geq \kappa/3$.

To prove Theorem 3.2, it remains to show that Inequality (A.3) and Equality (A.4) hold. The former is a direct consequence of Lemma A.2; since if $\Delta \leq \kappa/3$, then $q' = q$, $\mathrm{SINR}_{\mathcal{A}'}(s_0, q) = \mathrm{SINR}_{\mathcal{A}}(s_0, q) = \beta$, and it follows that $\max\{\mathrm{dist}(s_0, p) \mid p \in \mathcal{H}'_0\}$ is realized at $p = q$. It remains to prove that $\delta' \leq \delta$ (Inequality (A.3) holds). We do so by showing that $B(s_0, \delta') \subseteq \mathcal{VH}_{\mathcal{A}}(s_0)$. We first show that $B(s_0, \delta') \subseteq \mathcal{H}_{\mathcal{A}}(s_0)$ Fix $\rho_i = \mathrm{dist}(s_i, q)$ for every $1 \leq i \leq n-1$. We argue that the ball $B(s_0, \delta')$ is strictly contained in the ball $B(q, \rho_i)$ for every $1 \leq i \leq n-1$. To see why this is true, observe that $-\Delta < 0 < \delta' = \mu'_r < a'_i$, hence the ball centered at $q = (-\Delta, 0, ..., 0)$ of radius $\rho_i = \Delta + a'_i$ strictly contains the ball of radius δ' centered at $s_0 = (0, ..., 0)$. Next, consider an arbitrary point $p \in B(s_0, \delta')$. We can now rewrite

$$\mathrm{dist}(s'_i, (\delta', 0, ..., 0)) = a'_i - \delta' = \min\{\mathrm{dist}(t, t') \mid t \in B(s_0, \delta'), t' \in \Phi B(q, \rho_i)\}$$

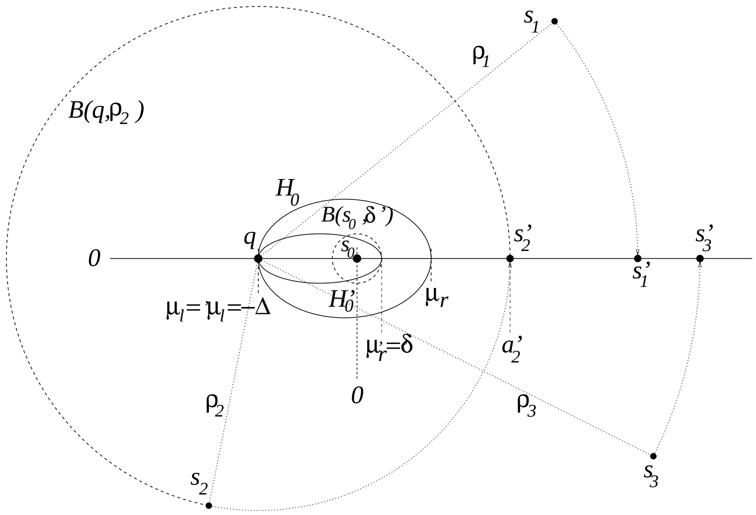

Figure 6: $\mathcal{A}'$ is obtained from $\mathcal{A}$ by relocating each station s_i on the x-axis.

for every $1 \leq i \leq n-1$. Recall that $s_i \in \Phi(B(q,\rho_i))$, thus $\text{dist}(s_i,p) \geq \text{dist}(s'_i,(\delta',0,...,0))$. Therefore $\text{I}_{\mathcal{A}}(s_0,p) \leq \text{I}_{\mathcal{A}'}(s_0,(\delta',0,...,0))$ and $\text{SINR}_{\mathcal{A}}(s_0,p) \geq \text{SINR}_{\mathcal{A}'}(s_0,(\delta',0)) = \beta$. It follows that $p \in \mathcal{H}_{\mathcal{A}}(s_0)$. Finally, it remains to show that $B(s_0,\delta')$ is in the Voronoi cell of s_0 in $\mathcal{A}$. Note that the transformation to $\mathcal{A}'$ cannot decrease the distance between the interfering stations and s_0, i.e., $\text{dist}(s_0,s_i) \geq \text{dist}(s_0,s'_i)$. Hence, $\kappa' \leq \kappa$. Since $\delta' \leq \kappa'/2$, it holds that also $\delta' \leq \kappa/2$, the claim follows. ∎